PNAS

Supporting Information for

Invariant dual mechanics of tensegrity and origami

Xiangxin Dang and Glaucio H. Paulino

Glaucio H. Paulino.
E-mail: gpaulino@princeton.edu

This PDF file includes:

Supporting text
Figs. S1 to S15
Legend for Movie S1
SI References

Other supporting materials for this manuscript include the following:

Movie S1

Supporting Information Text

A. Equilibrium of prismatic tensegrities. Standard prismatic tensegrities are composed of m bars, $2m$ horizontal cables, and m side cables (Fig. S1). There are in total $2m$ nodes, which are indexed counterclockwise. The nodes $1, ..., m$ are vertices of the bottom polygon while the nodes $m+1, ..., 2m$ belong to the top polygon. In particular, side cables connects node i and node $i+m$, $i = 1, ..., m$. The top and bottom polygons are the same in their size, but twisted by a angle α. A prismatic tensegrity exhibits dihedral symmetry, i.e., m-fold rotations and m 2-fold rotations. Thus, the equilibrium can be expressed by the force balance at one node. Under the coordinate system in Fig. S1B, the unit direction vectors of members at the node 2 can be expressed by

$$\mathbf{n}_{a1} = \left[-\sin\frac{\pi}{m}, -\cos\frac{\pi}{m}, 0\right], \qquad [1]$$

$$\mathbf{n}_{a2} = \left[-\sin\frac{\pi}{m}, \cos\frac{\pi}{m}, 0\right], \qquad [2]$$

$$\mathbf{n}_{b} = \left[\frac{r}{b}(\cos\alpha - 1), \frac{r}{b}\sin\alpha, \frac{h}{b}\right], \qquad [3]$$

$$\mathbf{n}_{c} = \left\{\frac{r}{c}\left[\cos\left(\alpha + \frac{2\pi}{m}\right) - 1\right], \frac{r}{c}\sin\left(\alpha + \frac{2\pi}{m}\right), \frac{h}{c}\right\}, \qquad [4]$$

where r is the circumferential radius of the base polygons and h is the height of the tensegrity. Specifically, the radius r can be expressed by

$$r = \frac{a}{2\sin\frac{\pi}{m}}. \qquad [5]$$

We denote the internal forces in the horizontal cables, the side cables, and the diagonal bars by F_a, F_b, and F_c, respectively; and their lengths by a, b, and c, respectively. Positive values of F_a, F_b, and F_c represent tension and negative values represent compression. The equilibrium at the node 2 can be expressed by

$$F_a\mathbf{n}_{a1} + F_a\mathbf{n}_{a2} + F_b\mathbf{n}_b + F_c\mathbf{n}_c = \mathbf{0}. \qquad [6]$$

It will be convenient to define the force densities, which are the forces per unit length of the members:

$$q_a = \frac{F_a}{a}, \quad q_b = \frac{F_b}{b}, \quad \text{and} \quad q_c = \frac{F_c}{c}. \qquad [7]$$

Then, the component form of Eq. (6) becomes

$$\begin{cases} -4q_a\sin^2\frac{\pi}{m} + q_b(\cos\alpha - 1) + q_c\left[\cos\left(\alpha + \frac{2\pi}{m}\right) - 1\right] = 0, \\ q_b\sin\alpha + q_c\sin\left(\alpha + \frac{2\pi}{m}\right) = 0, \\ q_b + q_c = 0. \end{cases} \qquad [8]$$

Equation (8) is solvable only if the twist angle satisfies:

$$\alpha = \frac{\pi}{2} - \frac{\pi}{m}. \qquad [9]$$

Under the condition of Eq. (9), we obtain the solutions of Eq. (8):

$$\frac{q_a}{-q_c} = \frac{1}{2\sin\frac{\pi}{m}} \quad \text{and} \quad \frac{q_b}{-q_c} = 1. \qquad [10]$$

Considering Eq. (7), Eq. (10) can be rewritten as

$$\frac{F_a}{-F_c} = \frac{a}{2c\sin\frac{\pi}{m}} \quad \text{and} \quad \frac{F_b}{-F_c} = \frac{b}{c}. \qquad [11]$$

B. Infinitesimal kinematics of Kresling origami. The geometry of Kresling origami can be determined by four independent parameters, m, r, h, and φ (Fig. S2). The first two are associated with the base polygon, i.e., its side number m and circumscribed radius r. The other two describe the global shape of the configuration, i.e., its height h and twist angle φ. We use a, b, and c to denote lengths of the horizontal edge, the side edge, and the diagonal edge, respectively. The corresponding dihedral angles are denoted by θ_a, θ_b, and θ_c, respectively. We stipulate that the dihedral angles θ_a, θ_b, and θ_c are defined over the interval $[0, 2\pi]$. A dihedral angle in $[0, \pi)$ corresponds to a valley crease and a dihedral angle in $(\pi, 2\pi]$ corresponds to a mountain crease. These edge lengths and dihedral angles can be expressed in terms of the four parameters (1):

$$a = 2r\sin\frac{\pi}{m}, \qquad [12]$$

$$b = \sqrt{h^2 + 4r^2\sin^2\frac{\varphi}{2}}, \qquad [13]$$

$$c = \sqrt{h^2 + 4r^2 \sin^2\left(\frac{\varphi}{2} + \frac{\pi}{m}\right)}, \tag{14}$$

$$\theta_a = 2\pi - \arctan\left[\frac{h}{2r\sin\left(\frac{\varphi}{2}+\frac{\pi}{m}\right)\sin\frac{\varphi}{2}}\right], \tag{15}$$

$$\theta_b = 2\pi - \arccos\left[\frac{4r^2\sin^2\left(\frac{\varphi}{2}+\frac{\pi}{m}\right)\sin^2\frac{\varphi}{2} - h^2\cos\left(\varphi+\frac{2\pi}{m}\right)}{4r^2\sin^2\left(\frac{\varphi}{2}+\frac{\pi}{m}\right)\sin^2\frac{\varphi}{2}+h^2}\right], \tag{16}$$

$$\theta_c = \arccos\left[\frac{4r^2\sin^2\left(\frac{\varphi}{2}+\frac{\pi}{m}\right)\sin^2\frac{\varphi}{2} - h^2\cos\varphi}{4r^2\sin^2\left(\frac{\varphi}{2}+\frac{\pi}{m}\right)\sin^2\frac{\varphi}{2}+h^2}\right]. \tag{17}$$

The polygon side number m is constant for given Kresling origami. To investigate the infinitesimal mechanisms of Kresling origami, we calculate the total derivatives of the edge lengths a, b, and c, with respect to the parameters r, h, and φ:

$$\mathrm{d}a = 2\sin\frac{\pi}{m}\mathrm{d}r, \tag{18}$$

$$\mathrm{d}b = \frac{h}{b}\mathrm{d}h + \frac{r^2}{b}\sin\varphi\mathrm{d}\varphi + \frac{4r}{b}\sin^2\frac{\varphi}{2}\mathrm{d}r, \tag{19}$$

$$\mathrm{d}c = \frac{h}{c}\mathrm{d}h + \frac{r^2}{c}\sin\left(\varphi+\frac{2\pi}{m}\right)\mathrm{d}\varphi + \frac{4r}{c}\sin^2\left(\frac{\varphi}{2}+\frac{\pi}{m}\right)\mathrm{d}r. \tag{20}$$

The infinitesimal mechanisms lead to zero first-order changes of edge lengths, that is,

$$\mathrm{d}a = \mathrm{d}b = \mathrm{d}c = 0. \tag{21}$$

We consider the Kresling origami that has the same geometry as the standard prismatic tensegrity. That is, Eq. (9) holds for the Kresling origami. Then, the initial twist angle φ_0 satisfies

$$\varphi_0 = \alpha = \frac{\pi}{2} - \frac{\pi}{m}. \tag{22}$$

Substituting Eqs. (21) and (22) into Eqs. (18)–(20), we obtain

$$\mathrm{d}r = 0 \quad \text{and} \quad \mathrm{d}h = -\frac{r^2\cos\frac{\pi}{m}}{h}\mathrm{d}\varphi. \tag{23}$$

Under the conditions given by Eqs. (22) and (23), we calculate the total derivative of the dihedral angles θ_a, θ_b, and θ_c, with respect to φ:

$$\mathrm{d}\theta_a = \frac{a}{2h\sin\frac{\pi}{m}}\mathrm{d}\varphi, \quad \mathrm{d}\theta_b = \frac{b}{h}\mathrm{d}\varphi, \quad \text{and} \quad \mathrm{d}\theta_c = -\frac{c}{h}\mathrm{d}\varphi. \tag{24}$$

Equations (10) and (24) lead to

$$q_a : q_b : q_c = \frac{\mathrm{d}\theta_a}{a} : \frac{\mathrm{d}\theta_b}{b} : \frac{\mathrm{d}\theta_c}{c}. \tag{25}$$

Considering Eq. (7), Eq. (25) is equivalent to

$$F_a : F_b : F_c = \mathrm{d}\theta_a : \mathrm{d}\theta_b : \mathrm{d}\theta_c. \tag{26}$$

C. Framework analysis preliminaries. We consider a nondegenerate three-dimensional linear transformation $\mathbf{T}$, which can be, for example, rotation, reflection, stretch, and shear. The transformation $\mathbf{T}$ is applied to nodal coordinates $\mathbf{r}_i = [x_i, y_i, z_i]^{\mathrm{T}}$ $(i = 1, 2, ..., N)$ of a three-dimensional free-standing pin-jointed framework. We do not consider degenerate linear transformations, such as projection, because we focus on frameworks in three-dimensional space. We will investigate the equilibrium and super-stability of frameworks under the transformation $\mathbf{T}$. *We do not change the connectivity matrix* $\mathbf{C}$ *(defined below) before and after the transformation.*

C.1. Linear transformation. We write the transformation $\mathbf{T}$ in the matrix form as

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \quad [27]$$

In addition, we define two high-dimensional transformation matrices:

$$\widetilde{\mathbf{T}}_{3M} = \begin{bmatrix} T_{11}\mathbf{I}_M & T_{12}\mathbf{I}_M & T_{13}\mathbf{I}_M \\ T_{21}\mathbf{I}_M & T_{22}\mathbf{I}_M & T_{23}\mathbf{I}_M \\ T_{31}\mathbf{I}_M & T_{32}\mathbf{I}_M & T_{33}\mathbf{I}_M \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{T}}_{3N} = \begin{bmatrix} T_{11}\mathbf{I}_N & T_{12}\mathbf{I}_N & T_{13}\mathbf{I}_N \\ T_{21}\mathbf{I}_N & T_{22}\mathbf{I}_N & T_{23}\mathbf{I}_N \\ T_{31}\mathbf{I}_N & T_{32}\mathbf{I}_N & T_{33}\mathbf{I}_N \end{bmatrix}, \quad [28]$$

where $\mathbf{I}_M$ is the M-by-M identity matrix, and $\mathbf{I}_N$ is the N-by-N identity matrix. For each joint i, the old coordinates x_i, y_i, z_i are transformed to the new coordinates $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_i$ by

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \\ \tilde{z}_i \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}. \quad [29]$$

Then, the new nodal coordinate vectors $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, $\tilde{\mathbf{z}}$ can be expressed as

$$\begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \\ \tilde{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{x} + T_{12}\mathbf{y} + T_{13}\mathbf{z} \\ T_{21}\mathbf{x} + T_{22}\mathbf{y} + T_{23}\mathbf{z} \\ T_{31}\mathbf{x} + T_{32}\mathbf{y} + T_{33}\mathbf{z} \end{bmatrix} = \mathbf{T}_{3N} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}, \quad [30]$$

where

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_N \end{bmatrix}, \ \tilde{\mathbf{y}} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_N \end{bmatrix}, \ \tilde{\mathbf{z}} = \begin{bmatrix} \tilde{z}_1 \\ \vdots \\ \tilde{z}_N \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \ \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}. \quad [31]$$

We recall the relationships between the coordinate difference vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and the nodal coordinate vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$:

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}. \quad [32]$$

The connectivity matrix $\mathbf{C}$ is of size $M \times N$, where M is the number of members (struts and cables) and N is the number of joints (nodes). Corresponding to the member k connecting joint i and j, the k-th row in $\mathbf{C}$ represents the connection:

$$C_{k,p} = \begin{cases} \operatorname{sign}(j-p), & \text{if } \ p = i; \\ \operatorname{sign}(i-p), & \text{if } \ p = j; \\ 0, & \text{otherwise}, \end{cases} \quad [33]$$

where $k = 1, 2, ..., M$ and $p = 1, 2, ..., N$. To obtain the new coordinate difference vectors $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$, $\tilde{\mathbf{w}}$, we first show the following relationship:

$$\mathbf{T}_{3M} \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{I}_M\mathbf{C} & T_{12}\mathbf{I}_M\mathbf{C} & T_{13}\mathbf{I}_M\mathbf{C} \\ T_{21}\mathbf{I}_M\mathbf{C} & T_{22}\mathbf{I}_M\mathbf{C} & T_{23}\mathbf{I}_M\mathbf{C} \\ T_{31}\mathbf{I}_M\mathbf{C} & T_{32}\mathbf{I}_M\mathbf{C} & T_{33}\mathbf{I}_M\mathbf{C} \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{C}\mathbf{I}_N & T_{12}\mathbf{C}\mathbf{I}_N & T_{13}\mathbf{C}\mathbf{I}_N \\ T_{21}\mathbf{C}\mathbf{I}_N & T_{22}\mathbf{C}\mathbf{I}_N & T_{23}\mathbf{C}\mathbf{I}_N \\ T_{31}\mathbf{C}\mathbf{I}_N & T_{32}\mathbf{C}\mathbf{I}_N & T_{33}\mathbf{C}\mathbf{I}_N \end{bmatrix} = \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix} \mathbf{T}_{3N}. \quad [34]$$

Then, we have

$$\begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{v}} \\ \tilde{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{y}} \\ \tilde{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix} \mathbf{T}_{3N} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \mathbf{T}_{3M} \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \mathbf{T}_{3M} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{u} + T_{12}\mathbf{v} + T_{13}\mathbf{w} \\ T_{21}\mathbf{u} + T_{22}\mathbf{v} + T_{23}\mathbf{w} \\ T_{31}\mathbf{u} + T_{32}\mathbf{v} + T_{33}\mathbf{w} \end{bmatrix}, \quad [35]$$

where

$$\tilde{\mathbf{u}} = \begin{bmatrix} \tilde{u}_1 \\ \vdots \\ \tilde{u}_M \end{bmatrix}, \ \tilde{\mathbf{v}} = \begin{bmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_M \end{bmatrix}, \ \tilde{\mathbf{w}} = \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_M \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_M \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_M \end{bmatrix}. \quad [36]$$

It is worth noting that, for each member k connecting joint i and j, Eq. (35) actually gives the following relationship:

$$\begin{bmatrix} \tilde{u}_k \\ \tilde{v}_k \\ \tilde{w}_k \end{bmatrix} = \operatorname{sign}(i-j) \begin{bmatrix} \tilde{x}_j - \tilde{x}_i \\ \tilde{y}_j - \tilde{y}_i \\ \tilde{z}_j - \tilde{z}_i \end{bmatrix} = \operatorname{sign}(i-j) \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} x_j - x_i \\ y_j - y_i \\ z_j - z_i \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix}. \quad [37]$$

Also, we have the diagonal version:

$$\begin{bmatrix} \widetilde{\mathbf{U}} \\ \widetilde{\mathbf{V}} \\ \widetilde{\mathbf{W}} \end{bmatrix} = \mathbf{T}_{3M} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{W} \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{I}_M & T_{12}\mathbf{I}_M & T_{13}\mathbf{I}_M \\ T_{21}\mathbf{I}_M & T_{22}\mathbf{I}_M & T_{23}\mathbf{I}_M \\ T_{31}\mathbf{I}_M & T_{32}\mathbf{I}_M & T_{33}\mathbf{I}_M \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{W} \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{U} + T_{12}\mathbf{V} + T_{13}\mathbf{W} \\ T_{21}\mathbf{U} + T_{22}\mathbf{V} + T_{23}\mathbf{W} \\ T_{31}\mathbf{U} + T_{32}\mathbf{V} + T_{33}\mathbf{W} \end{bmatrix}, \quad [38]$$

where

$$\widetilde{\mathbf{U}} = \mathrm{diag}(\tilde{\mathbf{u}}),\ \widetilde{\mathbf{V}} = \mathrm{diag}(\tilde{\mathbf{v}}),\ \widetilde{\mathbf{W}} = \mathrm{diag}(\tilde{\mathbf{w}}),\ \mathbf{U} = \mathrm{diag}(\mathbf{u}),\ \mathbf{V} = \mathrm{diag}(\mathbf{v}),\ \mathbf{W} = \mathrm{diag}(\mathbf{w}). \quad [39]$$

We denote the old and new member length vectors by $\mathbf{l}$ and $\tilde{\mathbf{l}}$, respectively. Their components, ℓ_k and $\tilde{\ell}_k$, satisfy

$$\ell_k^2 = u_k^2 + v_k^2 + w_k^2 \quad \text{and} \quad \tilde{\ell}_k^2 = \tilde{u}_k^2 + \tilde{v}_k^2 + \tilde{w}_k^2. \quad [40]$$

Finally, we give the diagonal versions of member lengths:

$$\mathbf{L} = \mathrm{diag}(\mathbf{l}) \quad \text{and} \quad \widetilde{\mathbf{L}} = \mathrm{diag}(\tilde{\mathbf{l}}). \quad [41]$$

C.2. Free-standing equilibrium condition (2). We use $\mathbf{s}$ to denote the member force vector of size $M \times 1$, of which the k-th component F_k is the internal force in the member k. The equilibrium matrix $\mathbf{D}$ is expressed by

$$\mathbf{D} = \begin{bmatrix} \mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{W} \end{bmatrix} \mathbf{L}^{-1}. \quad [42]$$

There is no external force applied to free-standing frameworks. Then, the equilibrium condition reads

$$\mathbf{D}\mathbf{s} = \mathbf{0}. \quad [43]$$

C.3. Kinematic indeterminacy condition. The kinematic matrix $\mathbf{B}$ is expressed by

$$\mathbf{B} = \mathbf{D}^{\mathrm{T}}. \quad [44]$$

We use $\mathbf{m}$ to denote the infinitesimal mechanism vector of size $3N \times 1$, of which the p-th, $(p+N)$-th, and $(p+2N)$-th components $\mathrm{d}x_p$, $\mathrm{d}y_p$, $\mathrm{d}z_p$ are the infinitesimal nodal displacements of the p-th node in the x, y, and z directions, respectively. The infinitesimal mechanism vector $\mathbf{m}$ satisfies

$$\mathbf{B}\mathbf{m} = \mathbf{0}. \quad [45]$$

C.4. Jacobian matrix. We aim to derive the infinitesimal dihedral angle variation vector $\mathrm{d}\boldsymbol{\theta}$ under the infinitesimal mechanism $\mathbf{m}$. To do this, we investigate the member j connecting node $\hat{k}$ and node $\hat{j}$, as shown in Fig. S4. We also use the index $\hat{k}\hat{j}$ to denote the member j, indicating the two nodes connected by the member. Notice that the framework is assumed to be triangulated. As a result, there exist the nodes $\hat{\imath}$ and $\hat{\ell}$, as well as the members $\hat{\imath}\hat{j}$, $\hat{\imath}\hat{k}$, $\hat{\ell}\hat{j}$, and $\hat{k}\hat{\ell}$, such that the member $\hat{k}\hat{j}$ is the common edge of the triangle $\hat{k}\hat{j}\hat{\imath}$ and the triangle $\hat{k}\hat{\ell}\hat{j}$. Suppose the j-th component in the vector $\boldsymbol{\theta}$, denoted by θ_j, is the dihedral angle between the triangles $\hat{k}\hat{j}\hat{\imath}$ and $\hat{k}\hat{\ell}\hat{j}$. We define the following relative position vectors

$$\mathbf{r}_{\hat{p}\hat{q}} = \mathbf{r}_{\hat{p}} - \mathbf{r}_{\hat{q}}, \quad \text{for} \quad \hat{p}, \hat{q} = \hat{\imath}, \hat{j}, \hat{k}, \hat{\ell}. \quad [46]$$

We use $\mathbf{n}_{\hat{\imath}}$ and $\mathbf{n}_{\hat{\ell}}$ to denote the normal vectors of the triangles $\hat{k}\hat{j}\hat{\imath}$ and $\hat{k}\hat{\ell}\hat{j}$, respectively. These normal vectors are defined by

$$\begin{aligned} \mathbf{n}_{\hat{\imath}} &= \mathbf{r}_{\hat{\imath}\hat{j}} \times \mathbf{r}_{\hat{k}\hat{j}}, \\ \mathbf{n}_{\hat{\ell}} &= \mathbf{r}_{\hat{k}\hat{j}} \times \mathbf{r}_{\hat{k}\hat{\ell}}. \end{aligned} \quad [47]$$

The explicit expression of θ_j is given by (3)

$$\theta_j = \eta \arccos\left(\frac{\mathbf{n}_{\hat{\imath}}}{\|\mathbf{n}_{\hat{\imath}}\|} \cdot \frac{\mathbf{n}_{\hat{\ell}}}{\|\mathbf{n}_{\hat{\ell}}\|} \right) \mod 2\pi, \quad [48]$$

where

$$\eta = \begin{cases} \mathrm{sign}(\mathbf{n}_{\hat{\imath}} \cdot \mathbf{r}_{\hat{k}\hat{\ell}}), & \mathbf{n}_{\hat{\imath}} \cdot \mathbf{r}_{\hat{k}\hat{\ell}} \neq 0, \\ 1, & \mathbf{n}_{\hat{\imath}} \cdot \mathbf{r}_{\hat{k}\hat{\ell}} = 0. \end{cases} \quad [49]$$

The symbol "mod" is the modulo operator. Equation (48) is well-defined such that θ_j represents all the possible rotations between the triangles $\hat{k}\hat{j}\hat{\imath}$ and $\hat{k}\hat{\ell}\hat{j}$ from 0 to 2π. The Jacobian of θ_j can be explicitly expressed as (3)

$$\begin{aligned} \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{\imath}}} &= \frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{\imath}}\|^2} \mathbf{n}_{\hat{\imath}}, \\ \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{\ell}}} &= -\frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{\ell}}\|^2} \mathbf{n}_{\hat{\ell}}, \\ \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{j}}} &= \left(\frac{\mathbf{r}_{\hat{\imath}\hat{j}} \cdot \mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} - 1 \right) \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{\imath}}} - \frac{\mathbf{r}_{\hat{k}\hat{\ell}} \cdot \mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{\ell}}}, \\ \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{k}}} &= \left(\frac{\mathbf{r}_{\hat{k}\hat{\ell}} \cdot \mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} - 1 \right) \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{\ell}}} - \frac{\mathbf{r}_{\hat{\imath}\hat{j}} \cdot \mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} \frac{\partial \theta_j}{\partial \mathbf{r}_{\hat{\imath}}}. \end{aligned} \quad [50]$$

We use $\mathbf{r}$ to denote the nodal coordinate vector of size $3N \times 1$, of which the p-th, $(p+N)$-th, and $(p+2N)$-th components x_p, y_p, z_p are the nodal coordinates of the p-th node. Equation (50) gives the components of the Jacobian $\mathbf{J}$ of $\boldsymbol{\theta}$ with respect to $\mathbf{r}$. Noticing that $\mathrm{d}\mathbf{r} = \mathbf{m}$, we have

$$\mathrm{d}\boldsymbol{\theta} = \mathbf{J}\mathbf{m}. \tag{51}$$

C.5. Super-stability definition and conditions (2). Definition of super-stability: *If a prestressed free-standing pin-jointed framework is always stable in the state of self-equilibrium, in the sense of having locally strict minimum of the total potential energy, irrespective of material properties as well as level of prestresses (i.e., signs of the prestresses should not be changed, while the magnitude of the prestresses can be arbitrarily scaled in proportion satisfying the self-equilibrium equations), then it is super-stable.* To investigate the super-stability, we need the force density matrix $\mathbf{E}$ and the geometry matrix $\mathbf{G}$.

We use $\mathbf{q}$ to denote force density vector of size $M \times 1$, of which the k-th component $q_k = F_k/\ell_k$ is the internal force density in the member k. Considering

$$\mathbf{s} = \mathbf{L}\mathbf{q}, \tag{52}$$

the equilibrium condition Eq. (43) is equivalent to

$$\mathbf{D}\mathbf{L}\mathbf{q} = \mathbf{0}. \tag{53}$$

The diagonal version of $\mathbf{q}$ is denoted by

$$\mathbf{Q} = \mathrm{diag}(\mathbf{q}). \tag{54}$$

We define the force density matrix by

$$\mathbf{E} = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}. \tag{55}$$

The equilibrium condition Eq. (43) can be equivalently expressed as

$$\begin{cases} \mathbf{E}\mathbf{x} = \mathbf{0}, \\ \mathbf{E}\mathbf{y} = \mathbf{0}, \\ \mathbf{E}\mathbf{z} = \mathbf{0}. \end{cases} \tag{56}$$

Equation (56) implies that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are eigenvectors of $\mathbf{E}$ with respect to the zero eigenvalue. Since translating a tensegrity does not change its self-equilibrium, $\mathbf{E}$ must have another eigenvector $\mathbf{i}$:

$$\mathbf{E}\mathbf{i} = \mathbf{0}, \tag{57}$$

where all the entries of $\mathbf{i}$ are 1. Equations (56) and (57) suggest that the force density matrix $\mathbf{E}$ must have the rank deficiency (number of zero eigenvalues) not less than four, so as to guarantee that the corresponding tensegrity is three-dimensional (i.e., $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are linearly independent).

The geometry matrix $\mathbf{G}$ is defined by

$$\mathbf{G} = \left[\mathbf{U}\mathbf{u}, \mathbf{V}\mathbf{v}, \mathbf{W}\mathbf{w}, \mathbf{U}\mathbf{v}, \mathbf{U}\mathbf{w}, \mathbf{V}\mathbf{w}\right]. \tag{58}$$

The geometry matrix $\mathbf{G}$ describes the non-trivial affine motions

$$\mathbf{d}_{\mathrm{a}}^{x} = \begin{bmatrix}\mathbf{x}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}, \quad \mathbf{d}_{\mathrm{a}}^{y} = \begin{bmatrix}\mathbf{0}\\ \mathbf{y}\\ \mathbf{0}\end{bmatrix}, \quad \mathbf{d}_{\mathrm{a}}^{z} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{z}\end{bmatrix}, \quad \mathbf{d}_{\mathrm{a}}^{xy} = \begin{bmatrix}\mathbf{y}\\ \mathbf{x}\\ \mathbf{0}\end{bmatrix}, \quad \mathbf{d}_{\mathrm{a}}^{xz} = \begin{bmatrix}\mathbf{z}\\ \mathbf{0}\\ \mathbf{x}\end{bmatrix}, \quad \mathbf{d}_{\mathrm{a}}^{yz} = \begin{bmatrix}\mathbf{0}\\ \mathbf{z}\\ \mathbf{y}\end{bmatrix}, \tag{59}$$

by the following relationship:

$$\mathbf{B}\left[\mathbf{d}_{\mathrm{a}}^{x}, \mathbf{d}_{\mathrm{a}}^{y}, \mathbf{d}_{\mathrm{a}}^{z}, \frac{\mathbf{d}_{\mathrm{a}}^{xy}}{2}, \frac{\mathbf{d}_{\mathrm{a}}^{xz}}{2}, \frac{\mathbf{d}_{\mathrm{a}}^{yz}}{2}\right] = \mathbf{L}^{-1}\mathbf{G}. \tag{60}$$

If the following three conditions are all satisfied, then the three-dimensional free-standing pin-jointed framework is super stable:

1. *Rank of the geometry matrix* $\mathbf{G}$ *is six;*

2. *The force density matrix* $\mathbf{E}$ *has the minimum necessary rank deficiency four;*

3. *The force density matrix* $\mathbf{E}$ *is positive semi-definite.*

Moreover, the first and third conditions are also necessary for super-stability.

D. Invariance theorems.

Definition 1. *A new free-standing pin-jointed framework is obtained by applying a nondegenerate three-dimensional linear transformation* $\mathbf{T}$ *to the nodal coordinates* $\mathbf{r}_k = [x_k, y_k, z_k]^{\mathrm{T}}$ $(k = 1, 2, ..., N)$ *of an old three-dimensional free-standing pin-jointed framework.*

Lemma 1. *If the old framework has* d_s *independent states of self-stress and* d_m *independent infinitesimal mechanisms, then the new framework must have the same number of independent states of self-stress* $\tilde{d}_s = d_s$ *and the same number of independent infinitesimal mechanisms* $\tilde{d}_m = d_m$.

Proof. We denote the old equilibrium matrix by $\mathbf{D}$. To obtain the new equilibrium matrix $\widetilde{\mathbf{D}}$ after the transformation, we need the transposed version of Eq. (34):

$$\begin{bmatrix}\mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C}\end{bmatrix}^{\mathrm{T}} \mathbf{T}_{3M} = \begin{bmatrix} T_{11}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M & T_{12}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M & T_{13}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M \\ T_{21}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M & T_{22}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M & T_{23}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M \\ T_{31}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M & T_{32}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M & T_{33}\mathbf{C}^{\mathrm{T}}\mathbf{I}_M \end{bmatrix} = \begin{bmatrix} T_{11}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} & T_{12}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} & T_{13}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} \\ T_{21}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} & T_{22}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} & T_{23}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} \\ T_{31}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} & T_{32}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} & T_{33}\mathbf{I}_N\mathbf{C}^{\mathrm{T}} \end{bmatrix} = \mathbf{T}_{3N} \begin{bmatrix}\mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C}\end{bmatrix}^{\mathrm{T}}. \quad [61]$$

Then, the new equilibrium matrix $\widetilde{\mathbf{D}}$ can be expressed as

$$\widetilde{\mathbf{D}} = \begin{bmatrix}\mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C}\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\widetilde{\mathbf{U}} \\ \widetilde{\mathbf{V}} \\ \widetilde{\mathbf{W}}\end{bmatrix} \widetilde{\mathbf{L}}^{-1} = \begin{bmatrix}\mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C}\end{bmatrix}^{\mathrm{T}} \mathbf{T}_{3M} \begin{bmatrix}\mathbf{U} \\ \mathbf{V} \\ \mathbf{W}\end{bmatrix} \widetilde{\mathbf{L}}^{-1} = \mathbf{T}_{3N} \begin{bmatrix}\mathbf{C} & & \\ & \mathbf{C} & \\ & & \mathbf{C}\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\mathbf{U} \\ \mathbf{V} \\ \mathbf{W}\end{bmatrix} \widetilde{\mathbf{L}}^{-1} = \mathbf{T}_{3N}\mathbf{D}(\mathbf{L}\widetilde{\mathbf{L}}^{-1}). \quad [62]$$

We can calculate the determinants:

$$\det(\mathbf{T}_{3N}) = [\det(\mathbf{T})]^N > 0, \quad [63]$$

$$\det(\mathbf{L}\widetilde{\mathbf{L}}^{-1}) = \prod_{k=1}^{M} \frac{\ell_k}{\tilde{\ell}_k} > 0. \quad [64]$$

In other words, both the matrices $\mathbf{T}_{3N}$ and $\mathbf{L}\widetilde{\mathbf{L}}^{-1}$ are full-rank. Therefore, we have

$$\mathrm{rank}(\widetilde{\mathbf{D}}) = \mathrm{rank}[\mathbf{T}_{3N}\mathbf{D}(\mathbf{L}\widetilde{\mathbf{L}}^{-1})] = \mathrm{rank}(\mathbf{D}), \quad [65]$$

which yields

$$\tilde{d}_s = M - \mathrm{rank}(\widetilde{\mathbf{D}}) = M - \mathrm{rank}(\mathbf{D}) = d_s. \quad [66]$$

Following Eq. (62), we can write the new kinematic matrix as

$$\widetilde{\mathbf{B}} = \widetilde{\mathbf{D}}^{\mathrm{T}} = (\mathbf{L}\widetilde{\mathbf{L}}^{-1})\mathbf{D}^{\mathrm{T}}\mathbf{T}_{3N}^{\mathrm{T}} = (\mathbf{L}\widetilde{\mathbf{L}}^{-1})\mathbf{B}\mathbf{T}_{3N}^{\mathrm{T}}. \quad [67]$$

Then, we have

$$\mathrm{rank}(\widetilde{\mathbf{B}}) = \mathrm{rank}[(\mathbf{L}\widetilde{\mathbf{L}}^{-1})\mathbf{B}\mathbf{T}_{3N}^{\mathrm{T}}] = \mathrm{rank}(\mathbf{B}), \quad [68]$$

which yields

$$\tilde{d}_m = 3N - 6 - \mathrm{rank}(\widetilde{\mathbf{B}}) = 3N - 6 - \mathrm{rank}(\mathbf{B}) = d_m. \quad [69]$$

□

Theorem 1 (tensegrity: invariance of equilibrium). *If the old framework has a state of self-stress with internal forces* F_i $(i = 1, 2, ..., M)$*, then the new framework must have a state of self-stress with internal forces* $\tilde{F}_i = (\tilde{\ell}_i/\ell_i)F_i$ $(i = 1, 2, ..., M)$.

Proof. Before the transformation, the equilibrium condition reads

$$\mathbf{D}\mathbf{L}\mathbf{q} = \mathbf{0}. \quad [70]$$

From Eq. (62), we know

$$\mathbf{D} = \mathbf{T}_{3N}^{-1}\widetilde{\mathbf{D}}(\widetilde{\mathbf{L}}\mathbf{L}^{-1}). \quad [71]$$

We substitute Eq. (71) into the old equilibrium condition Eq. (70), and obtain

$$\mathbf{D}\mathbf{L}\mathbf{q} = \mathbf{T}_{3N}^{-1}\widetilde{\mathbf{D}}(\widetilde{\mathbf{L}}\mathbf{L}^{-1})\mathbf{L}\mathbf{q} = \mathbf{T}_{3N}^{-1}\widetilde{\mathbf{D}}\widetilde{\mathbf{L}}\mathbf{q} = \mathbf{0}. \quad [72]$$

Since $\mathbf{T}_{3N}^{-1}$ is full-rank, we have

$$\widetilde{\mathbf{D}}\widetilde{\mathbf{L}}\mathbf{q} = \mathbf{0}. \quad [73]$$

The old force density vector $\mathbf{q}$ satisfies the new equilibrium condition. Therefore, the new framework has a state of self-stress with the force density vector

$$\tilde{\mathbf{q}} = \mathbf{q}. \quad [74]$$

Equation (74) is equivalent to

$$\widetilde{\mathbf{L}}^{-1}\tilde{\mathbf{s}} = \mathbf{L}^{-1}\mathbf{s}. \quad [75]$$

The component form of Eq. (75) is $\tilde{F}_i = (\tilde{\ell}_i/\ell_i)F_i$ $(i = 1, 2, ..., M)$. □

Theorem 2 (origami: invariance of infinitesimal mechanism). *If the old framework has an infinitesimal mechanism with displacements* $\mathrm{d}\mathbf{r}_k = [\mathrm{d}x_k, \mathrm{d}y_k, \mathrm{d}z_k]^{\mathrm{T}}$ *($k = 1, 2, ..., N$) and dihedral angle variations* $\mathrm{d}\theta_j$ *($j = 1, 2, ..., M$), then the new framework must have an infinitesimal mechanism with displacements* $\mathrm{d}\tilde{\mathbf{r}}_k = \det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}\mathbf{r}_k$ *and dihedral angle variations* $\mathrm{d}\tilde{\theta}_j = (\tilde{\ell}_j/\ell_j)\mathrm{d}\theta_j$.

Proof. The proof includes two parts. First, we show that $\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}\mathbf{r}_k$ are displacements of an infinitesimal mechanism of the new framework. We write the old mechanism $\mathbf{m}$ in the matrix form as

$$\mathbf{m} = \begin{bmatrix} \mathrm{d}\mathbf{x} \\ \mathrm{d}\mathbf{y} \\ \mathrm{d}\mathbf{z} \end{bmatrix}, \quad \text{where} \ \ \mathrm{d}\mathbf{x} = \begin{bmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_N \end{bmatrix}, \mathrm{d}\mathbf{y} = \begin{bmatrix} \mathrm{d}y_1 \\ \vdots \\ \mathrm{d}y_N \end{bmatrix}, \mathrm{d}\mathbf{z} = \begin{bmatrix} \mathrm{d}z_1 \\ \vdots \\ \mathrm{d}z_N \end{bmatrix}. \tag{76}$$

The old mechanism $\mathbf{m}$ satisfies

$$\mathbf{B}\mathbf{m} = \mathbf{D}^{\mathrm{T}}\mathbf{m} = \mathbf{0}. \tag{77}$$

We substitute Eq. (71) into the old compatibility condition Eq. (77), and obtain

$$\mathbf{D}^{\mathrm{T}}\mathbf{m} = (\mathbf{L}^{-1}\widetilde{\mathbf{L}})\widetilde{\mathbf{D}}^{\mathrm{T}}\mathbf{T}_{3N}^{-\mathrm{T}}\mathbf{m} = (\mathbf{L}^{-1}\widetilde{\mathbf{L}})\widetilde{\mathbf{B}}\mathbf{T}_{3N}^{-\mathrm{T}}\mathbf{m} = \mathbf{0}. \tag{78}$$

Since $\mathbf{L}^{-1}\widetilde{\mathbf{L}}$ is full-rank, we have

$$\widetilde{\mathbf{B}}\mathbf{T}_{3N}^{-\mathrm{T}}\mathbf{m} = \mathbf{0}. \tag{79}$$

Considering that $\mathbf{T}$ is nondegenerate, Eq. (79) is equivalent to

$$\widetilde{\mathbf{B}}[\det(\mathbf{T})\mathbf{T}_{3N}^{-\mathrm{T}}\mathbf{m}] = \mathbf{0}. \tag{80}$$

Therefore, the new framework has the following mechanism

$$\tilde{\mathbf{m}} = \det(\mathbf{T})\mathbf{T}_{3N}^{-\mathrm{T}}\mathbf{m}. \tag{81}$$

Equation (81) is equivalent to

$$\frac{1}{\det(\mathbf{T})}\mathbf{T}_{3N}^{\mathrm{T}}\tilde{\mathbf{m}} = \mathbf{m}. \tag{82}$$

We write the new mechanism $\tilde{\mathbf{m}}$ in the matrix form as

$$\tilde{\mathbf{m}} = \begin{bmatrix} \mathrm{d}\tilde{\mathbf{x}} \\ \mathrm{d}\tilde{\mathbf{y}} \\ \mathrm{d}\tilde{\mathbf{z}} \end{bmatrix}, \quad \text{where} \ \ \mathrm{d}\tilde{\mathbf{x}} = \begin{bmatrix} \mathrm{d}\tilde{x}_1 \\ \vdots \\ \mathrm{d}\tilde{x}_N \end{bmatrix}, \mathrm{d}\tilde{\mathbf{y}} = \begin{bmatrix} \mathrm{d}\tilde{y}_1 \\ \vdots \\ \mathrm{d}\tilde{y}_N \end{bmatrix}, \mathrm{d}\tilde{\mathbf{z}} = \begin{bmatrix} \mathrm{d}\tilde{z}_1 \\ \vdots \\ \mathrm{d}\tilde{z}_N \end{bmatrix}. \tag{83}$$

We recall the matrix form of $\mathbf{T}_{3N}^{\mathrm{T}}$ in Eq. (28). Then, Eq. (82) can be rewritten as

$$\frac{1}{\det(\mathbf{T})}\begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}\begin{bmatrix} \mathrm{d}\tilde{x}_k \\ \mathrm{d}\tilde{y}_k \\ \mathrm{d}\tilde{z}_k \end{bmatrix} = \begin{bmatrix} \mathrm{d}x_k \\ \mathrm{d}y_k \\ \mathrm{d}z_k \end{bmatrix}, \tag{84}$$

or in the compact form as

$$\frac{1}{\det(\mathbf{T})}\mathbf{T}^{\mathrm{T}}\mathrm{d}\tilde{\mathbf{r}}_k = \mathrm{d}\mathbf{r}_k, \tag{85}$$

where $\mathrm{d}\mathbf{r}_k$ and $\mathrm{d}\tilde{\mathbf{r}}_k$ are respectively the old and the new infinitesimal mechanism displacements at node k, for $k = 1, 2, ..., N$. Since $\mathbf{T}$ is nondegenerate, the new framework has an infinitesimal mechanism with displacements

$$\boxed{\mathrm{d}\tilde{\mathbf{r}}_k = \det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}\mathbf{r}_k.} \tag{86}$$

The second part of the proof derives the new dihedral angle variation $\mathrm{d}\tilde{\theta}_j$ under the infinitesimal nodal displacements $\mathrm{d}\tilde{\mathbf{r}}_k$. To do this, we start by investigating the old framework. We use the notations in Section C.4 and Fig. S4. Given Eq. (50), the total differential $\mathrm{d}\theta_j$ can be calculated by

$$\begin{aligned}\mathrm{d}\theta_j =& \frac{\partial\theta_j}{\partial\mathbf{r}_{\hat{i}}}\cdot\mathrm{d}\mathbf{r}_{\hat{i}} + \frac{\partial\theta_j}{\partial\mathbf{r}_{\hat{\ell}}}\cdot\mathrm{d}\mathbf{r}_{\hat{\ell}} + \frac{\partial\theta_j}{\partial\mathbf{r}_{\hat{j}}}\cdot\mathrm{d}\mathbf{r}_{\hat{j}} + \frac{\partial\theta_j}{\partial\mathbf{r}_{\hat{k}}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}} \\ =& \frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{i}}\|^2}\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{i}} - \frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{\ell}}\|^2}\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{\ell}} + \left(\frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} - 1\right)\frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{i}}\|^2}\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{j}} + \frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{\ell}}\|^2}\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{j}} \\ &- \left(\frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} - 1\right)\frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{\ell}}\|^2}\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}} - \frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\|\mathbf{r}_{\hat{k}\hat{j}}\|}{\|\mathbf{n}_{\hat{i}}\|^2}\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}}.\end{aligned} \tag{87}$$

From Eq. (87), we can obtain

$$\begin{aligned}\frac{\mathrm{d}\theta_j}{\|\mathbf{r}_{\hat{k}\hat{j}}\|} &= \frac{\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{i}}}{\|\mathbf{n}_{\hat{i}}\|^2} - \frac{\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{\ell}}}{\|\mathbf{n}_{\hat{\ell}}\|^2} + \left(\frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} - 1\right)\frac{\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{j}}}{\|\mathbf{n}_{\hat{i}}\|^2} + \frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{j}}}{\|\mathbf{n}_{\hat{\ell}}\|^2} - \left(\frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2} - 1\right)\frac{\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}}}{\|\mathbf{n}_{\hat{\ell}}\|^2} - \frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}}}{\|\mathbf{n}_{\hat{i}}\|^2} \\ &= \frac{\mathbf{n}_{\hat{i}}\cdot(\mathrm{d}\mathbf{r}_{\hat{i}} - \mathrm{d}\mathbf{r}_{\hat{j}})}{\|\mathbf{n}_{\hat{i}}\|^2} + \frac{\mathbf{n}_{\hat{\ell}}\cdot(\mathrm{d}\mathbf{r}_{\hat{k}} - \mathrm{d}\mathbf{r}_{\hat{\ell}})}{\|\mathbf{n}_{\hat{\ell}}\|^2} + \frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\mathbf{n}_{\hat{i}}\cdot(\mathrm{d}\mathbf{r}_{\hat{j}} - \mathrm{d}\mathbf{r}_{\hat{k}})}{\|\mathbf{n}_{\hat{i}}\|^2} + \frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\mathbf{n}_{\hat{\ell}}\cdot(\mathrm{d}\mathbf{r}_{\hat{j}} - \mathrm{d}\mathbf{r}_{\hat{k}})}{\|\mathbf{n}_{\hat{\ell}}\|^2}.\end{aligned} \quad [88]$$

We define the following relative displacement vectors

$$\mathrm{d}\mathbf{r}_{\hat{p}\hat{q}} = \mathrm{d}\mathbf{r}_{\hat{p}} - \mathrm{d}\mathbf{r}_{\hat{q}}, \quad \text{for} \quad \hat{p},\hat{q} = \hat{i},\hat{j},\hat{k},\hat{\ell}. \quad [89]$$

Then, Eq. (88) becomes

$$\begin{aligned}\frac{\mathrm{d}\theta_j}{\|\mathbf{r}_{\hat{k}\hat{j}}\|} &= \frac{\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{i}\hat{j}}}{\|\mathbf{n}_{\hat{i}}\|^2} + \frac{\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}\hat{\ell}}}{\|\mathbf{n}_{\hat{\ell}}\|^2} - \frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\mathbf{n}_{\hat{i}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{n}_{\hat{i}}\|^2} - \frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\frac{\mathbf{n}_{\hat{\ell}}\cdot\mathrm{d}\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{n}_{\hat{\ell}}\|^2} \\ &= \left(\mathrm{d}\mathbf{r}_{\hat{i}\hat{j}} - \frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\mathrm{d}\mathbf{r}_{\hat{k}\hat{j}}\right)\cdot\frac{\mathbf{n}_{\hat{i}}}{\|\mathbf{n}_{\hat{i}}\|^2} + \left(\mathrm{d}\mathbf{r}_{\hat{k}\hat{\ell}} - \frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\mathrm{d}\mathbf{r}_{\hat{k}\hat{j}}\right)\cdot\frac{\mathbf{n}_{\hat{\ell}}}{\|\mathbf{n}_{\hat{\ell}}\|^2}.\end{aligned} \quad [90]$$

Since the infinitesimal mechanism does not change length of the members, we have

$$\mathrm{d}\mathbf{r}_{\hat{p}\hat{q}}\cdot\mathbf{r}_{\hat{p}\hat{q}} = 0, \quad \text{for} \quad \hat{p},\hat{q} = \hat{i},\hat{j},\hat{k},\hat{\ell}. \quad [91]$$

The motion of the triangle $\hat{k}\hat{j}\hat{i}$ relative to the node $\hat{j}$ is an infinitesimal rotation. According to Euler's rotation theorem (4), there must exist an angular velocity $\boldsymbol{\omega}$ that determines the infinitesimal rotation, such that

$$\boxed{\boldsymbol{\omega}\times\mathbf{r}_{\hat{i}\hat{j}} = \mathrm{d}\mathbf{r}_{\hat{i}\hat{j}} \quad \text{and} \quad \boldsymbol{\omega}\times\mathbf{r}_{\hat{k}\hat{j}} = \mathrm{d}\mathbf{r}_{\hat{k}\hat{j}}.} \quad [92]$$

If we look at the motion of the nodes $\hat{i}$, $\hat{j}$, $\hat{k}$, and $\hat{\ell}$ in the frame of reference of the triangle $\hat{k}\hat{j}\hat{i}$, the nodal displacements become

$$\begin{aligned}\mathrm{d}'\mathbf{r}_{\hat{i}\hat{j}} &= \mathrm{d}\mathbf{r}_{\hat{i}\hat{j}} - \boldsymbol{\omega}\times\mathbf{r}_{\hat{i}\hat{j}} = \mathbf{0}, \\ \mathrm{d}'\mathbf{r}_{\hat{k}\hat{j}} &= \mathrm{d}\mathbf{r}_{\hat{k}\hat{j}} - \boldsymbol{\omega}\times\mathbf{r}_{\hat{k}\hat{j}} = \mathbf{0}, \\ \mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}} &= \mathrm{d}\mathbf{r}_{\hat{k}\hat{\ell}} - \boldsymbol{\omega}\times\mathbf{r}_{\hat{k}\hat{\ell}}, \\ \mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}} &= \mathrm{d}'\mathbf{r}_{\hat{k}\hat{j}} - \mathrm{d}'\mathbf{r}_{\hat{\ell}\hat{j}} = -\mathrm{d}\mathbf{r}_{\hat{\ell}\hat{j}} + \boldsymbol{\omega}\times\mathbf{r}_{\hat{\ell}\hat{j}}.\end{aligned} \quad [93]$$

The choice of frame of reference does not affect the dihedral angle variation between the triangles $\hat{k}\hat{j}\hat{i}$ and $\hat{k}\hat{\ell}\hat{j}$. Therefore, Eq. (90) can be written as

$$\boxed{\frac{\mathrm{d}\theta_j}{\|\mathbf{r}_{\hat{k}\hat{j}}\|} = \left(\mathrm{d}'\mathbf{r}_{\hat{i}\hat{j}} - \frac{\mathbf{r}_{\hat{i}\hat{j}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\mathrm{d}'\mathbf{r}_{\hat{k}\hat{j}}\right)\cdot\frac{\mathbf{n}_{\hat{i}}}{\|\mathbf{n}_{\hat{i}}\|^2} + \left(\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}} - \frac{\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}}{\|\mathbf{r}_{\hat{k}\hat{j}}\|^2}\mathrm{d}'\mathbf{r}_{\hat{k}\hat{j}}\right)\cdot\frac{\mathbf{n}_{\hat{\ell}}}{\|\mathbf{n}_{\hat{\ell}}\|^2} = \frac{\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{n}_{\hat{\ell}}}{\|\mathbf{n}_{\hat{\ell}}\|^2}.} \quad [94]$$

After the transformation $\mathbf{T}$, the nodal position vectors become

$$\tilde{\mathbf{r}}_k = \mathbf{T}\mathbf{r}_k, \quad \text{for} \quad k = \hat{i},\hat{j},\hat{k},\hat{\ell}, \quad [95]$$

and the nodal infinitesimal displacements become

$$\mathrm{d}\tilde{\mathbf{r}}_k = \det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}\mathbf{r}_k, \quad \text{for} \quad k = \hat{i},\hat{j},\hat{k},\hat{\ell}. \quad [96]$$

The new relative nodal position vectors become

$$\tilde{\mathbf{r}}_{\hat{p}\hat{q}} = \mathbf{T}\mathbf{r}_{\hat{p}\hat{q}}, \quad \text{for} \quad \hat{p},\hat{q} = \hat{i},\hat{j},\hat{k},\hat{\ell}, \quad [97]$$

and the new relative displacement vectors become

$$\mathrm{d}\tilde{\mathbf{r}}_{\hat{p}\hat{q}} = \mathrm{d}\tilde{\mathbf{r}}_{\hat{p}} - \mathrm{d}\tilde{\mathbf{r}}_{\hat{q}} = \det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}\mathbf{r}_{\hat{p}\hat{q}}, \quad \text{for} \quad \hat{p},\hat{q} = \hat{i},\hat{j},\hat{k},\hat{\ell}. \quad [98]$$

The new mechanism can also be verified by the following derivation

$$\mathrm{d}\tilde{\mathbf{r}}_{\hat{p}\hat{q}}\cdot\tilde{\mathbf{r}}_{\hat{p}\hat{q}} = [\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}\mathbf{r}_{\hat{p}\hat{q}}]\cdot(\mathbf{T}\mathbf{r}_{\hat{p}\hat{q}}) = \det(\mathbf{T})(\mathrm{d}\mathbf{r}_{\hat{p}\hat{q}})^{\mathrm{T}}(\mathbf{T}^{-1}\mathbf{T})\mathbf{r}_{\hat{p}\hat{q}} = \det(\mathbf{T})\mathrm{d}\mathbf{r}_{\hat{p}\hat{q}}\cdot\mathbf{r}_{\hat{p}\hat{q}} = 0, \quad \text{for} \quad \hat{p},\hat{q} = \hat{i},\hat{j},\hat{k},\hat{\ell}. \quad [99]$$

We can directly write the new angular velocity vector $\tilde{\boldsymbol{\omega}}$ as

$$\tilde{\boldsymbol{\omega}} = \mathbf{T}\boldsymbol{\omega}. \quad [100]$$

We can immediately obtain the following relationship

$$\boxed{\begin{aligned}\tilde{\boldsymbol{\omega}}\times\tilde{\mathbf{r}}_{\hat{i}\hat{j}}&=(\mathbf{T}\boldsymbol{\omega})\times(\mathbf{T}\mathbf{r}_{\hat{i}\hat{j}})=\det(\mathbf{T})\mathbf{T}^{-1}(\boldsymbol{\omega}\times\mathbf{r}_{\hat{i}\hat{j}})=\det(\mathbf{T})\mathbf{T}^{-1}\mathrm{d}\mathbf{r}_{\hat{i}\hat{j}}=\mathrm{d}\tilde{\mathbf{r}}_{\hat{i}\hat{j}},\\ \tilde{\boldsymbol{\omega}}\times\tilde{\mathbf{r}}_{\hat{k}\hat{j}}&=(\mathbf{T}\boldsymbol{\omega})\times(\mathbf{T}\mathbf{r}_{\hat{k}\hat{j}})=\det(\mathbf{T})\mathbf{T}^{-1}(\boldsymbol{\omega}\times\mathbf{r}_{\hat{k}\hat{j}})=\det(\mathbf{T})\mathbf{T}^{-1}\mathrm{d}\mathbf{r}_{\hat{k}\hat{j}}=\mathrm{d}\tilde{\mathbf{r}}_{\hat{k}\hat{j}}.\end{aligned}} \tag{101}$$

Therefore, for the new framework, if we look at the motion of the nodes $\hat{i}$, $\hat{j}$, $\hat{k}$, and $\hat{\ell}$ in a frame of reference of the triangle $\hat{k}\hat{j}\hat{i}$, the nodal displacements become

$$\begin{aligned}\mathrm{d}'\tilde{\mathbf{r}}_{\hat{i}\hat{j}}&=\mathrm{d}\tilde{\mathbf{r}}_{\hat{i}\hat{j}}-\tilde{\boldsymbol{\omega}}\times\tilde{\mathbf{r}}_{\hat{i}\hat{j}}=\mathbf{0},\\ \mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{j}}&=\mathrm{d}\tilde{\mathbf{r}}_{\hat{k}\hat{j}}-\tilde{\boldsymbol{\omega}}\times\tilde{\mathbf{r}}_{\hat{k}\hat{j}}=\mathbf{0},\\ \mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}}&=\mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{j}}-\mathrm{d}'\tilde{\mathbf{r}}_{\hat{\ell}\hat{j}}=-\mathrm{d}\tilde{\mathbf{r}}_{\hat{\ell}\hat{j}}+\tilde{\boldsymbol{\omega}}\times\tilde{\mathbf{r}}_{\hat{\ell}\hat{j}}.\end{aligned} \tag{102}$$

The new normal vectors are given by

$$\begin{aligned}\tilde{\mathbf{n}}_{\hat{i}}&=\tilde{\mathbf{r}}_{\hat{i}\hat{j}}\times\tilde{\mathbf{r}}_{\hat{k}\hat{j}},\\ \tilde{\mathbf{n}}_{\hat{\ell}}&=\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\times\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}}.\end{aligned} \tag{103}$$

Now we can write the ratio of the new dihedral angle variation $\mathrm{d}\tilde{\theta}_j$ over the new member length $\tilde{\mathbf{r}}_{\hat{k}\hat{j}}$ as

$$\frac{\mathrm{d}\tilde{\theta}_j}{\|\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\|}=\left(\mathrm{d}'\tilde{\mathbf{r}}_{\hat{i}\hat{j}}-\frac{\tilde{\mathbf{r}}_{\hat{i}\hat{j}}\cdot\tilde{\mathbf{r}}_{\hat{k}\hat{j}}}{\|\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\|^2}\mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\right)\cdot\frac{\tilde{\mathbf{n}}_{\hat{i}}}{\|\tilde{\mathbf{n}}_{\hat{i}}\|^2}+\left(\mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}}-\frac{\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}}\cdot\tilde{\mathbf{r}}_{\hat{k}\hat{j}}}{\|\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\|^2}\mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\right)\cdot\frac{\tilde{\mathbf{n}}_{\hat{\ell}}}{\|\tilde{\mathbf{n}}_{\hat{\ell}}\|^2}=\frac{\mathrm{d}'\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}}\cdot\tilde{\mathbf{n}}_{\hat{\ell}}}{\|\tilde{\mathbf{n}}_{\hat{\ell}}\|^2}. \tag{104}$$

For any two vectors $\mathbf{a}$ and $\mathbf{b}$ and a linear transformation $\mathbf{M}$, the following identity holds true:

$$(\mathbf{M}\mathbf{a})\times(\mathbf{M}\mathbf{b})=\det(\mathbf{M})\mathbf{M}^{-\mathrm{T}}(\mathbf{a}\times\mathbf{b}). \tag{105}$$

Therefore, the new normal vectors can be rewritten as

$$\begin{aligned}\tilde{\mathbf{n}}_{\hat{i}}&=(\mathbf{T}\tilde{\mathbf{r}}_{\hat{i}\hat{j}})\times(\mathbf{T}\tilde{\mathbf{r}}_{\hat{k}\hat{j}})=\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}(\tilde{\mathbf{r}}_{\hat{i}\hat{j}}\times\tilde{\mathbf{r}}_{\hat{k}\hat{j}})=\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{i}},\\ \tilde{\mathbf{n}}_{\hat{\ell}}&=(\mathbf{T}\tilde{\mathbf{r}}_{\hat{k}\hat{j}})\times(\mathbf{T}\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}})=\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}(\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\times\tilde{\mathbf{r}}_{\hat{k}\hat{\ell}})=\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{\ell}}.\end{aligned} \tag{106}$$

Substituting Eqs. (98) and (106) into Eq. (104) yields

$$\boxed{\frac{\mathrm{d}\tilde{\theta}_j}{\|\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\|}=\frac{[\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}]\cdot[\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{\ell}}]}{\|\det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{\ell}}\|^2}=\frac{(\mathbf{T}^{-\mathrm{T}}\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}})\cdot(\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{\ell}})}{\|\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{\ell}}\|^2}.} \tag{107}$$

To find the relationship between $\mathrm{d}\theta_j/\|\mathbf{r}_{\hat{k}\hat{j}}\|$ and $\mathrm{d}\tilde{\theta}_j/\|\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\|$, we investigate the two vectors $\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}$ and $\mathbf{n}_{\hat{\ell}}$. From Eqs. (91) and (93), first, we have

$$\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}}=\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\cdot(\mathbf{r}_{\hat{k}\hat{\ell}}-\mathbf{r}_{\hat{j}\hat{\ell}})=(-\mathrm{d}\mathbf{r}_{\hat{\ell}\hat{j}}+\tilde{\boldsymbol{\omega}}\times\mathbf{r}_{\hat{\ell}\hat{j}})\cdot\mathbf{r}_{\hat{\ell}\hat{j}}=0. \tag{108}$$

Second, we have

$$\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{\ell}}=0. \tag{109}$$

For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the following identity holds true:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \tag{110}$$

Then, Eqs. (108) and (109) yield

$$\boxed{\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\times\mathbf{n}_{\hat{\ell}}=\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\times(\mathbf{r}_{\hat{k}\hat{j}}\times\mathbf{r}_{\hat{k}\hat{\ell}})=\mathbf{r}_{\hat{k}\hat{j}}(\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{\ell}})-\mathbf{r}_{\hat{k}\hat{\ell}}(\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}\cdot\mathbf{r}_{\hat{k}\hat{j}})=0.} \tag{111}$$

Therefore, the vectors $\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}$ and $\mathbf{n}_{\hat{\ell}}$ are collinear. From Eq. (94), we obtain the following relationship

$$\boxed{\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}=\frac{\mathrm{d}\theta_j}{\|\mathbf{r}_{\hat{k}\hat{j}}\|}\mathbf{n}_{\hat{\ell}},} \tag{112}$$

The linear transformation $\mathbf{T}^{-\mathrm{T}}$ preserves the collinearity, that is

$$\boxed{\mathbf{T}^{-\mathrm{T}}\mathrm{d}'\mathbf{r}_{\hat{k}\hat{\ell}}=\frac{\mathrm{d}\theta_j}{\|\mathbf{r}_{\hat{k}\hat{j}}\|}(\mathbf{T}^{-\mathrm{T}}\mathbf{n}_{\hat{\ell}}).} \tag{113}$$

Equations (107) and (113) yield

$$\boxed{\frac{\mathrm{d}\theta_j}{\|\mathbf{r}_{\hat{k}\hat{j}}\|}=\frac{\mathrm{d}\tilde{\theta}_j}{\|\tilde{\mathbf{r}}_{\hat{k}\hat{j}}\|}.} \tag{114}$$

Equation (114) holds true for the dihedral angle variation at each member $j = 1, 2, ..., M$. This result can also be written in the following compact form

$$\boxed{\mathbf{L}^{-1}\mathbf{J}\mathbf{m} = \widetilde{\mathbf{L}}^{-1}\tilde{\mathbf{J}}\tilde{\mathbf{m}},} \tag{115}$$

where $\mathbf{J}$ is the old Jacobian matrix of the dihedral angle θ_j $(j = 1, 2, ..., M)$ with respect to the nodal coordinates x_k, y_k, z_k $(k = 1, 2, ..., N)$, and $\tilde{\mathbf{J}}$ is the new Jacobian matrix of the dihedral angle $\tilde{\theta}_j$ $(j = 1, 2, ..., M)$ with respect to the nodal coordinates $\tilde{x}_k, \tilde{y}_k, \tilde{z}_k$ $(k = 1, 2, ..., N)$. □

Lemma 2. *The rank of the geometry matrix* $\mathbf{G}$ *is the same for the old and new frameworks.*

Proof. Before the transformation, the old geometry matrix $\mathbf{G}$ has the form of Eq. (58). After the transformation, we write the new geometry matrix $\widetilde{\mathbf{G}}$ as

$$\widetilde{\mathbf{G}} = \left[\widetilde{\mathbf{U}}\tilde{\mathbf{u}}, \widetilde{\mathbf{V}}\tilde{\mathbf{v}}, \widetilde{\mathbf{W}}\tilde{\mathbf{w}}, \widetilde{\mathbf{U}}\tilde{\mathbf{v}}, \widetilde{\mathbf{U}}\tilde{\mathbf{w}}, \widetilde{\mathbf{V}}\tilde{\mathbf{w}}\right]. \tag{116}$$

Inserting Eqs. (35) and (38) into Eq. (116), we obtain

$$\widetilde{\mathbf{G}} = \mathbf{G}\mathbf{T}_{\mathrm{G}}, \tag{117}$$

where $\mathbf{T}_{\mathrm{G}}$ is defined by

$$\mathbf{T}_{\mathrm{G}} = \begin{bmatrix} T_{11}^2 & T_{21}^2 & T_{31}^2 & T_{11}T_{21} & T_{11}T_{31} & T_{21}T_{31} \\ T_{12}^2 & T_{22}^2 & T_{32}^2 & T_{12}T_{22} & T_{12}T_{32} & T_{22}T_{32} \\ T_{13}^2 & T_{23}^2 & T_{33}^2 & T_{13}T_{23} & T_{13}T_{33} & T_{23}T_{33} \\ 2T_{11}T_{12} & 2T_{21}T_{22} & 2T_{31}T_{32} & T_{11}T_{22}+T_{12}T_{21} & T_{11}T_{32}+T_{12}T_{31} & T_{21}T_{32}+T_{22}T_{31} \\ 2T_{11}T_{13} & 2T_{21}T_{23} & 2T_{31}T_{33} & T_{11}T_{23}+T_{13}T_{21} & T_{11}T_{33}+T_{13}T_{31} & T_{21}T_{33}+T_{23}T_{31} \\ 2T_{12}T_{13} & 2T_{22}T_{23} & 2T_{32}T_{33} & T_{12}T_{23}+T_{13}T_{22} & T_{12}T_{33}+T_{13}T_{32} & T_{22}T_{33}+T_{23}T_{32} \end{bmatrix}. \tag{118}$$

To obtain Eq. (117), we have considered that $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ are diagonal matrices, which satisfy the following relationships

$$\mathbf{V}\mathbf{u} = \mathbf{U}\mathbf{v}, \;\; \mathbf{W}\mathbf{u} = \mathbf{U}\mathbf{w}, \;\; \mathbf{W}\mathbf{v} = \mathbf{V}\mathbf{w}. \tag{119}$$

We can calculate the determinate of $\mathbf{T}_{\mathrm{G}}$:

$$\det(\mathbf{T}_{\mathrm{G}}) = [\det(\mathbf{T})]^4, \tag{120}$$

where

$$\det(\mathbf{T}) = T_{11}T_{22}T_{33} - T_{11}T_{23}T_{32} - T_{12}T_{21}T_{33} + T_{12}T_{23}T_{31} + T_{13}T_{21}T_{32} - T_{13}T_{22}T_{31}. \tag{121}$$

Since $\mathbf{T}$ is nondegenerate, we have

$$\det(\mathbf{T}) > 0 \quad \text{and} \quad \det(\mathbf{T}_{\mathrm{G}}) > 0. \tag{122}$$

In other words, the 6-by-6 matrix $\mathbf{T}_{\mathrm{G}}$ is full-rank. As a result, we have

$$\mathrm{rank}(\widetilde{\mathbf{G}}) = \mathrm{rank}(\mathbf{G}\mathbf{T}_{\mathrm{G}}) = \mathrm{rank}(\mathbf{G}). \tag{123}$$

□

Theorem 3 (tensegrity: invariance of super-stability). *If the old framework is super-stable and its force density matrix* $\mathbf{E}$ *is of rank deficiency four, then the new framework must be super-stable.*

Proof. To investigate the super-stability of the free-standing pin-jointed framework after the transformation $\mathbf{T}$, we check the three conditions in Section C.5. The first and third conditions are necessary for super-stability, so they are satisfied before the transformation. The second condition is also satisfied before the transformation, as stated in the theorem. We check the three conditions for the free-standing pin-jointed framework after the transformation. According to Lemma 2, the rank of geometry matrix $\mathbf{G}$ does not change. Thus, the first condition is satisfied. Next, according to Theorem 1, the force density vector $\mathbf{q}$, or equivalently its diagonal form $\mathbf{Q} = \mathrm{diag}(\mathbf{q})$, does not change. As a result, the force density matrix $\mathbf{E} = \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}$ does not change as well, and the second and the third conditions are satisfied. We have verified all the three conditions, so the free-standing pin-jointed framework is super-stable after the transformation. □

E. Rhombic truncated regular polyhedral (TRP) tensegrity. A rhombic TRP tensegrity is in self-equilibrium and super-stable *if and only if* the force densities satisfy (5):

$$\begin{aligned} -\frac{q_a}{q_c} &= Q_1, \\ -\frac{q_b}{q_c} &= \frac{\sqrt{\left(3Q_1^2 - 4Q_1 + 1\right)^2 + 2Q_1\left(6Q_1^2 - 7Q_1 + 2\right)\left(1 - \cos\frac{2\pi}{p}\right)} - \left(3Q_1^2 - 4Q_1 + 1\right)}{2\left(2Q_1 - 1\right)\left(1 - \cos\frac{2\pi}{p}\right)}, \end{aligned} \tag{124}$$

where $Q_1 > 1/2$ and $p = 3, 4, 5$ for the truncated tetrahedral, cubic, and dodecahedral configurations, respectively.

F. Generalization to projective transformations. An abstract proof of the projective invariance can be found in (6). Here, we provide additional derivations and discussions. For any node of a framework, let its Cartesian coordinates be $\mathbf{r} = [x, y, z]^{\mathrm{T}}$ and its homogeneous coordinates be $\mathbf{p} = [\mathbf{r}^{\mathrm{T}}, 1]^{\mathrm{T}} = [x, y, z, 1]^{\mathrm{T}}$. A projective transformation (or homography) acts as a nonlinear transformation on the Cartesian coordinates $\mathbf{r}$ and, equivalently, as a linear transformation on the homogeneous coordinates $\mathbf{p}$. The latter can be expressed as a 4-by-4 matrix:

$$\mathbf{H} = \begin{bmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{h}^{\mathrm{T}} & h \end{bmatrix}, \tag{125}$$

where $\mathbf{T}$ is a 3×3 linear transformation, $\mathbf{t}$ a translation vector, $\mathbf{h}$ a projection vector, and h a scale factor. Affine transformations, consisting of a linear transformation and a translation, are a special case of projective transformations with $\mathbf{h} = \mathbf{0}$. Additionally, if $\mathbf{t} = \mathbf{0}$, then $\mathbf{H}$ reduces to a linear transformation. For the duality problem, translations are immaterial, as they do not affect objective vectors such as forces, angular velocities, or displacements.

Applying the projective transformation gives

$$\tilde{\mathbf{p}} = \mathbf{H}\mathbf{p} = \begin{bmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{h}^{\mathrm{T}} & h \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{T}\mathbf{r} + \mathbf{t} \\ \mathbf{h}\cdot\mathbf{r} + h \end{bmatrix}, \tag{126}$$

so that, if $\mathbf{h}\cdot\mathbf{r} + h \neq \mathbf{0}$, the transformed Cartesian coordinates are

$$\tilde{\mathbf{r}} = \frac{\mathbf{T}\mathbf{r} + \mathbf{t}}{\mathbf{h}\cdot\mathbf{r} + h}. \tag{127}$$

If $\mathbf{h}\cdot\mathbf{r} + h = \mathbf{0}$, the node is transformed to infinity. We define the force density $\lambda_i = F_i/\ell_i$ for each member ij at a node j. The balance law at j reads

$$\sum_{i=1}^{n} [\lambda_i(\mathbf{r}_i - \mathbf{r}_j)] = \mathbf{0}. \tag{128}$$

After applying $\mathbf{H}$, the new force densities are given by (6):

$$\tilde{\lambda}_i = \lambda_i(\mathbf{h}\cdot\mathbf{r}_i + h)(\mathbf{h}\cdot\mathbf{r}_j + h). \tag{129}$$

Given Eq. (128), the new balance law can be derived as

$$\begin{aligned}
\sum_{i=1}^{n} \tilde{\lambda}_i(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j) &= \sum_{i=1}^{n} \left[\lambda_i(\mathbf{h}\cdot\mathbf{r}_i + h)(\mathbf{h}\cdot\mathbf{r}_j + h)\left(\frac{\mathbf{T}\mathbf{r}_i + \mathbf{t}}{\mathbf{h}\cdot\mathbf{r}_i + h} - \frac{\mathbf{T}\mathbf{r}_j + \mathbf{t}}{\mathbf{h}\cdot\mathbf{r}_j + h}\right)\right] \\
&= \sum_{i=1}^{n} [\lambda_i(\mathbf{h}\cdot\mathbf{r}_j + h)(\mathbf{T}\mathbf{r}_i + \mathbf{t}) - \lambda_i(\mathbf{h}\cdot\mathbf{r}_i + h)(\mathbf{T}\mathbf{r}_j + \mathbf{t})] \\
&= \mathbf{T}\sum_{i=1}^{n} \{\lambda_i[(\mathbf{h}\cdot\mathbf{r}_j)\mathbf{r}_i - (\mathbf{h}\cdot\mathbf{r}_i)\mathbf{r}_j]\} - \mathbf{t}\left\{\mathbf{h}\cdot\sum_{i=1}^{n}[\lambda_i(\mathbf{r}_i - \mathbf{r}_j)]\right\} + h\mathbf{T}\sum_{i=1}^{n}[\lambda_i(\mathbf{r}_i - \mathbf{r}_j)] \\
&= \mathbf{T}\sum_{i=1}^{n}[\mathbf{h}\times(\lambda_i\mathbf{r}_i\times\mathbf{r}_j)] - \mathbf{t}\left\{\mathbf{h}\cdot\sum_{i=1}^{n}[\lambda_i(\mathbf{r}_i - \mathbf{r}_j)]\right\} + h\mathbf{T}\sum_{i=1}^{n}[\lambda_i(\mathbf{r}_i - \mathbf{r}_j)] \\
&= \mathbf{T}\left\{\mathbf{h}\times\left[\left(\sum_{i=1}^{n}\lambda_i\right)\mathbf{r}_j\times\mathbf{r}_j\right]\right\} \\
&= \mathbf{0}.
\end{aligned} \tag{130}$$

This means projective transformations preserve static indeterminacy of tensegrity (cable-strut) frameworks. In another respect, if λ_i denotes infinitesimal dihedral angle variations per length, i.e., $\mathrm{d}\theta_i/\ell_i$, Eqs. (128)–(130) prove that projective transformations preserve kinematic indeterminacy of origami (panel-hinge) frameworks.

Regarding tensegrity frameworks, we note a crucial distinction between a general nonlinear projective transformation (with perspective, i.e., $\mathbf{h} \neq \mathbf{0}$) and a linear transformation (which preserves parallelism). That is, the nonlinear effect may alter the class number (e.g., from class-1 to class-n) of a tensegrity. These occur because the factors $(\mathbf{h}\cdot\mathbf{r}_i + h)(\mathbf{h}\cdot\mathbf{r}_j + h)$ in the transformed force densities $\tilde{\lambda}_i$ (in Eq. (129)) can vary in sign or vanish to zero across members. Negative factors reverse the compression or tension status in certain struts or cables and may make two or more struts meet at one node (i.e., change the class number from class-1 to class-n, $n \geq 1$). Moreover, zero factors transform corresponding nodes to infinity and cause physically invalid result, which should be avoided when applying the projective transformation.

Projective transformations are used to generate the irregular configurations in Fig. 6 in the Main Text. For Fig. 6A, $\mathbf{h} = [0, 0, 1]^{\mathrm{T}}$, and the transformation matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{131}$$

For Fig. 6B, $\mathbf{h} = [1, 0, 0]^{\mathrm{T}}$, and the transformation matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \qquad [132]$$

Since the effect of nonlinear projective transformations is dependent on the nodal coordinates, we give the related information as follows. For the regular prismatic configuration, the ratio of the height to the base radius is 1.5, and the valley crease (strut) length is scaled to 1. The Cartesian coordinate system is located at the centroid the structure, with x and y axes parallel to the base polygons and z axis perpendicular to the base polygons.

References

1. S Zang, D Misseroni, T Zhao, GH Paulino, Kresling origami mechanics explained: Experiments and theory. *J. Mech. Phys. Solids* **188**, 105630 (2024).
2. J Zhang, M Ohsaki, *Tensegrity Structures.* (Springer), (2015).
3. K Liu, GH Paulino, Nonlinear mechanics of non-rigid origami: An efficient computational approach. *Proc. Royal Soc. A: Math. Phys. Eng. Sci.* **473**, 20170348 (2017).
4. B Palais, R Palais, S Rodi, A disorienting look at Euler's theorem on the axis of a rotation. *The Am. Math. Mon.* **116**, 892–909 (2009).
5. LY Zhang, Y Li, YP Cao, XQ Feng, A unified solution for self-equilibrium and super-stability of rhombic truncated regular polyhedral tensegrities. *Int. J. Solids Struct.* **50**, 234–245 (2013).
6. H Crapo, W Whiteley, Statics of frameworks and motions of panel structures: A projective geometric introduction. *Struct. Topol. 1982, núm. 6* (1982).

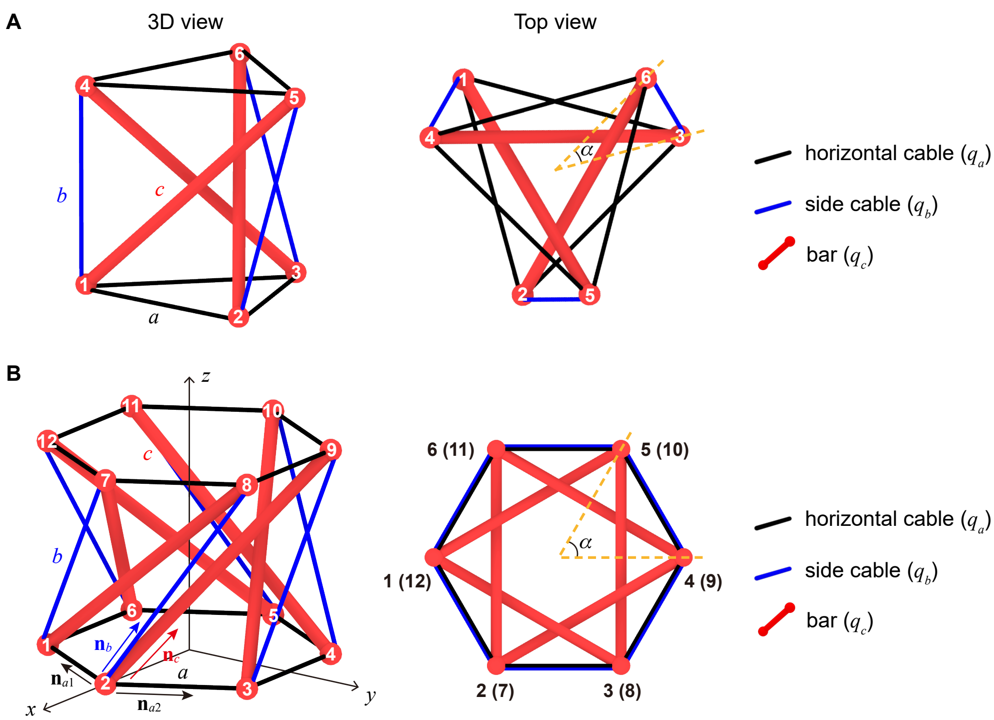

Fig. S1. Geometry and connectivity of prismatic tensegrities with (A) triangular and (B) hexagonal base polygons.

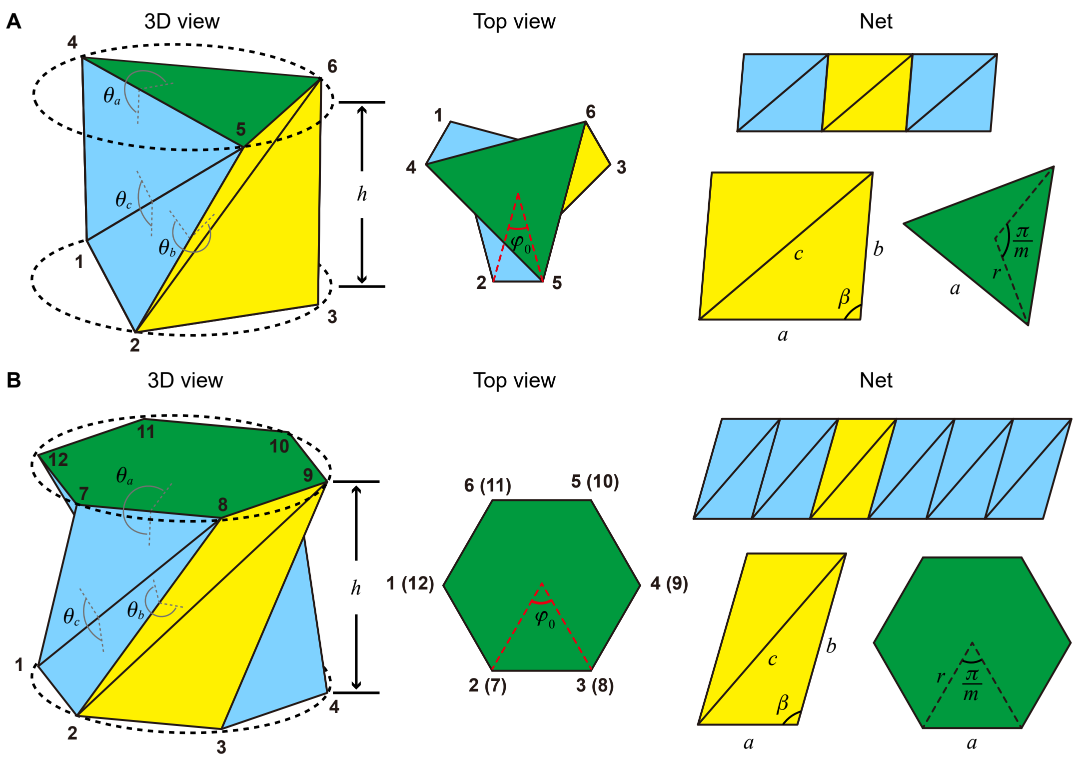

Fig. S2. Geometry and flat net of prismatic origami with (A) triangular and (B) hexagonal base polygons.

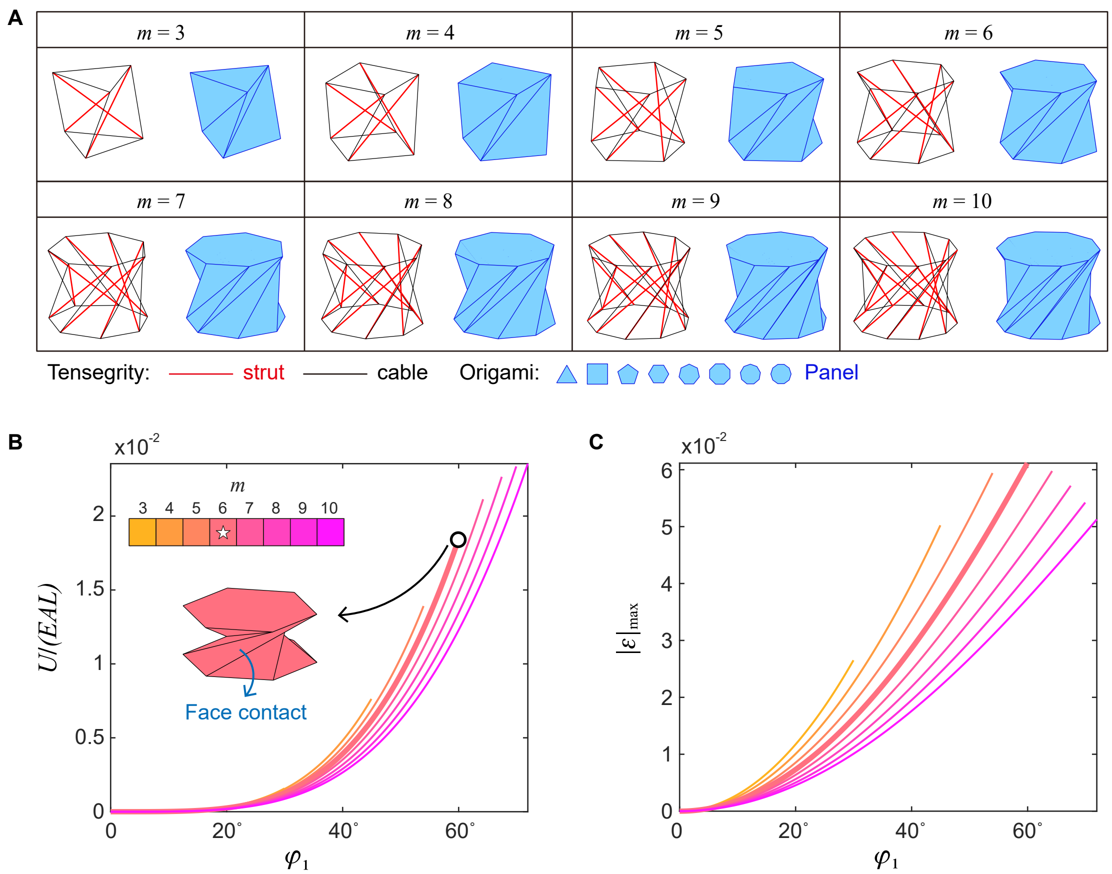

Fig. S3. Prismatic tensegrities and origami. (A) Dual configurations with m-gons. Simulation curves of (B) the nondimensional energy $U/(EAL)$ and (C) the maximum edge strain $|\varepsilon|_{\max}$ of these origami structures. The simulations are terminated when face contact occurs.

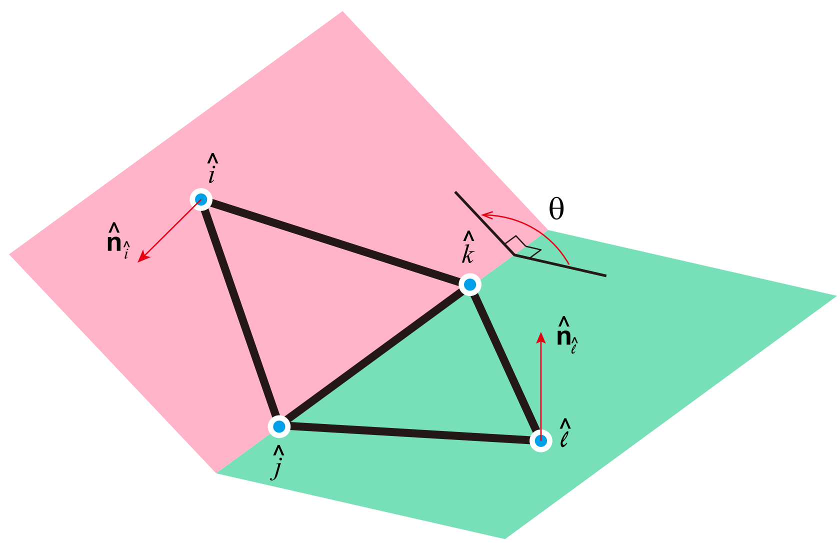

Fig. S4. Geometry of a dihedral angle between two triangular frames.

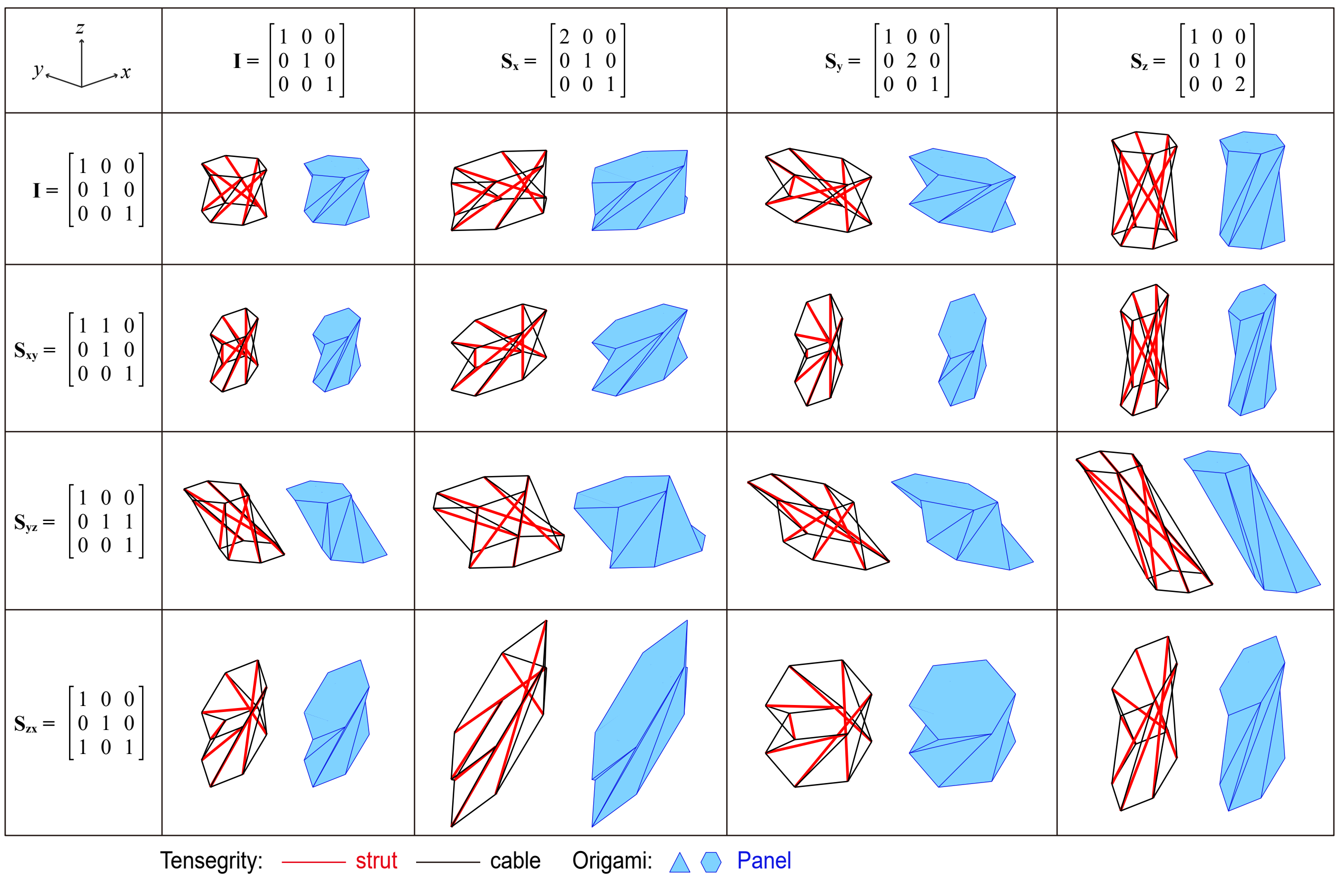

Fig. S5. Prismatic tensegrities and the dual origami obtained through a linear transformation $\mathbf{T}$ applied to the regular configurations (Top Left). The transformation $\mathbf{T}$ can be an identity ($\mathbf{I}$), a stretching ($\mathbf{S}_\mathrm{x}$, $\mathbf{S}_\mathrm{y}$, $\mathbf{S}_\mathrm{z}$), a shear ($\mathbf{S}_\mathrm{xy}$, $\mathbf{S}_\mathrm{yz}$, $\mathbf{S}_\mathrm{zx}$), and their combinations ($\mathbf{S}_\mathrm{xy}\mathbf{S}_\mathrm{z}$, etc).

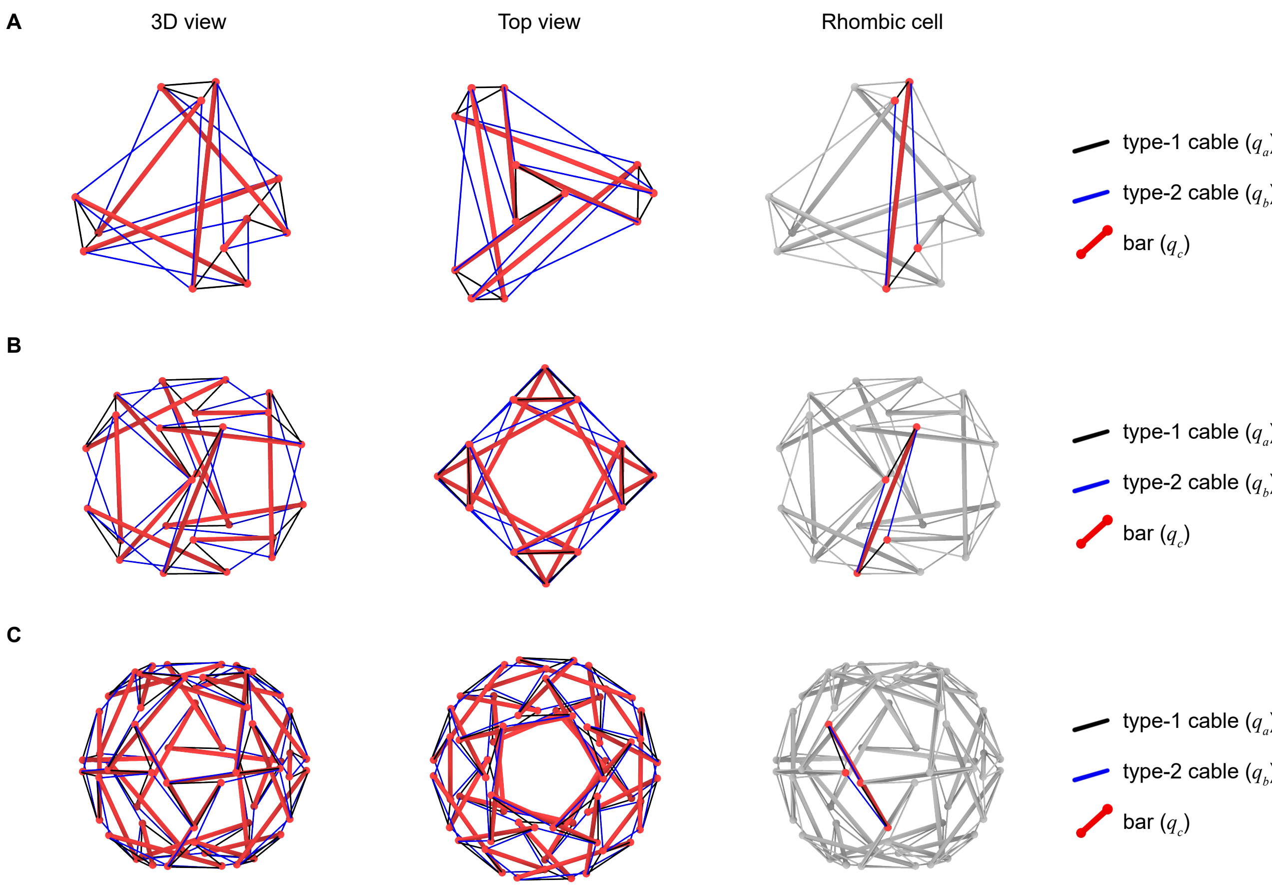

Fig. S6. Geometry and connectivity of rhombic truncated regular polyhedral (TRP) tensegrities with (A) tetrahedral, (B) cubic, and (C) dodecahedral configurations.

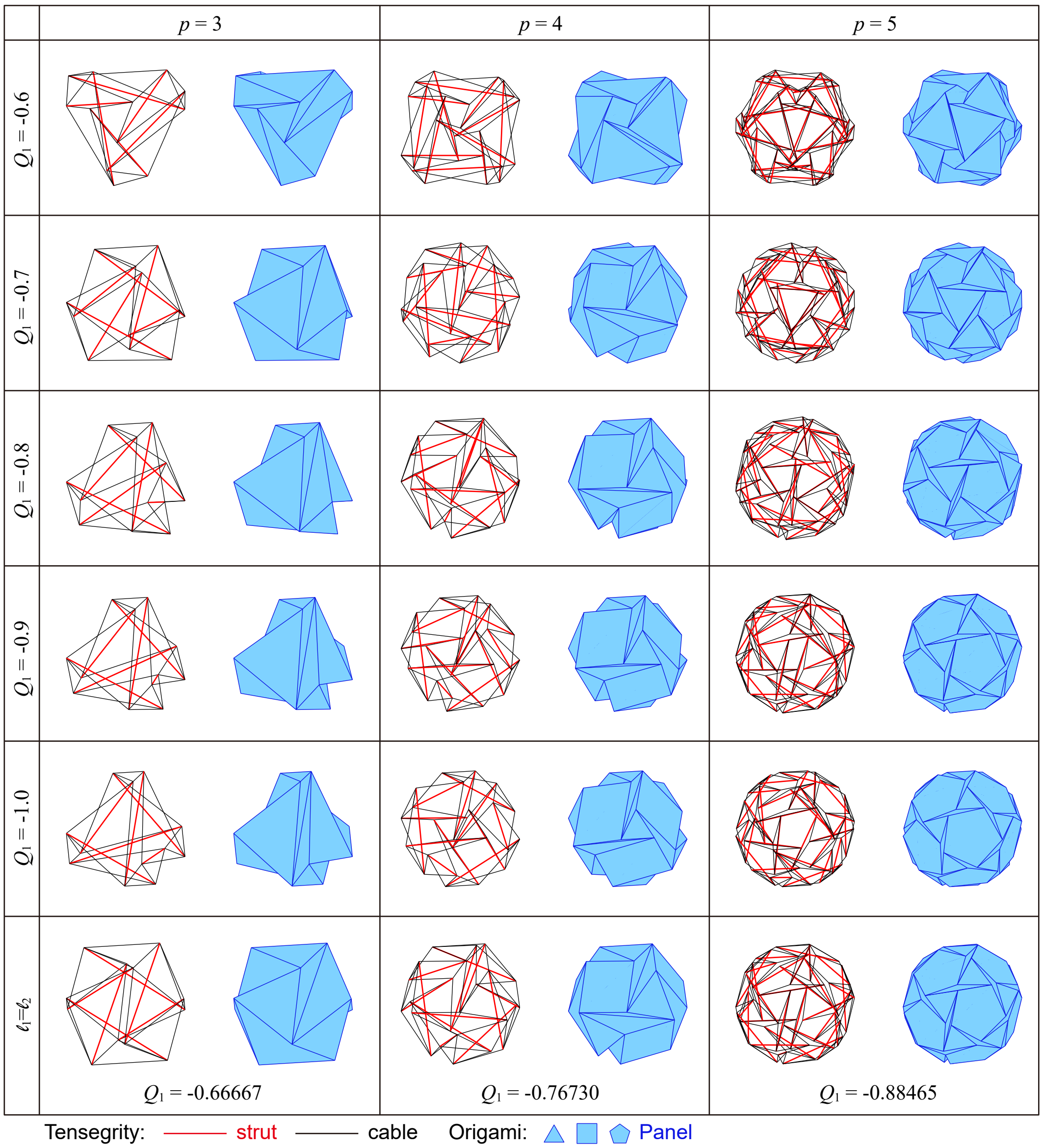

Fig. S7. Rhombic truncated regular polyhedral (TRP) tensegrities and the dual origami corresponding to different connectivities ($p = 3$ for tetrahedral, $p = 4$ for cubic, and $p = 5$ for dodecahedral configurations) and different force densities ($Q_1 = -q_a/q_c$; see q_a and q_c in Fig. S6). The bottom row shows configurations with identical cable lengths ($\ell_1 = \ell_2$).

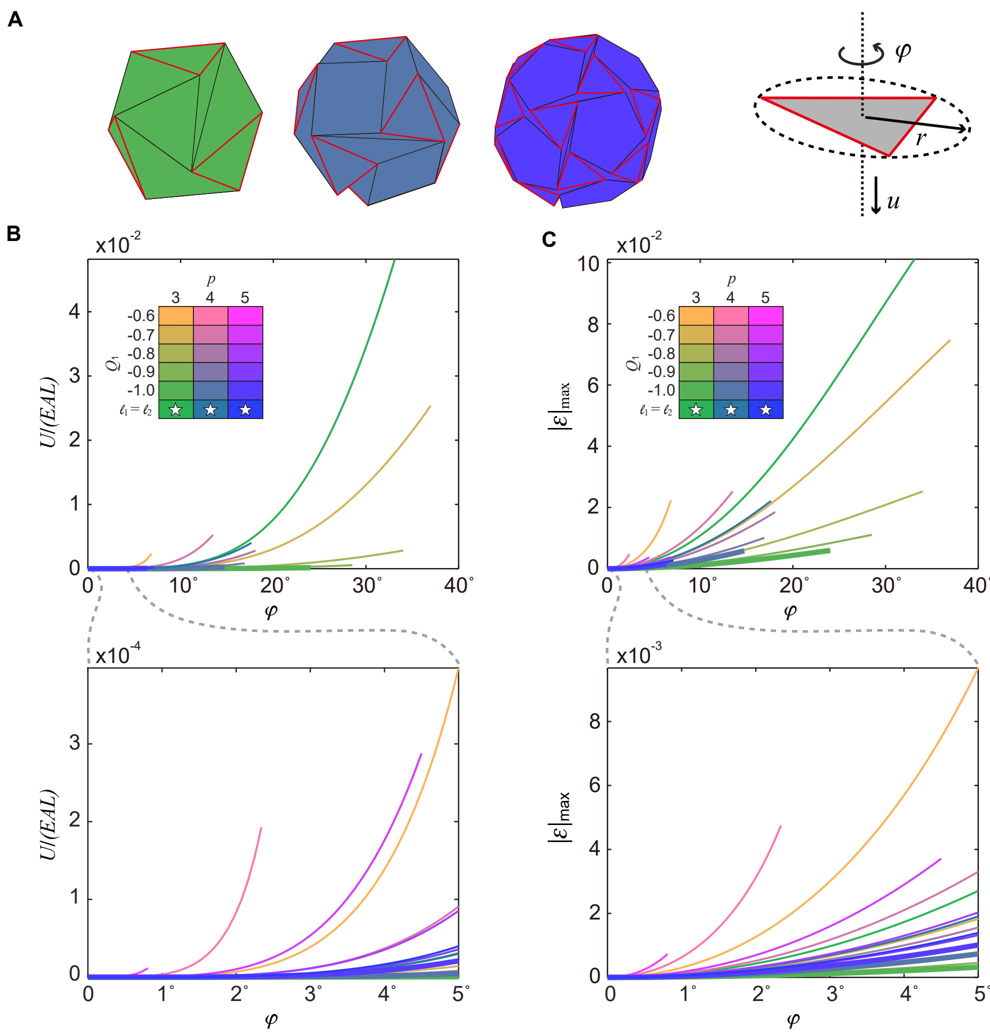

Fig. S8. Finite deformation simulation of regular polyhedral origami structures. (A) The finite deformations are simulated under the principle of minimum potential energy. Leveraging the configuration symmetry, the twist angle φ, circumferential radius r, and the translation u of base triangles define the overall deformation. (B) The nondimensional energy $U/(EAL)$ and (C) the maximum edge strain $|\varepsilon|_{\max}$ of the origami with different connectivities and different force densities in their dual tensegrities. The colormap is consistent with Fig. S7. The simulations are terminated when face contact occurs.

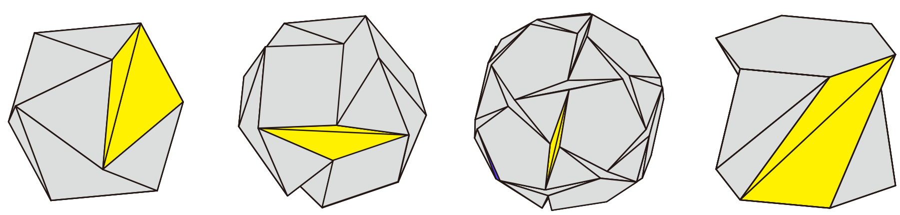

Fig. S9. The rhombic cells in the TRP tensegrities (Leftmost Three) and the Kresling origami (Far Right).

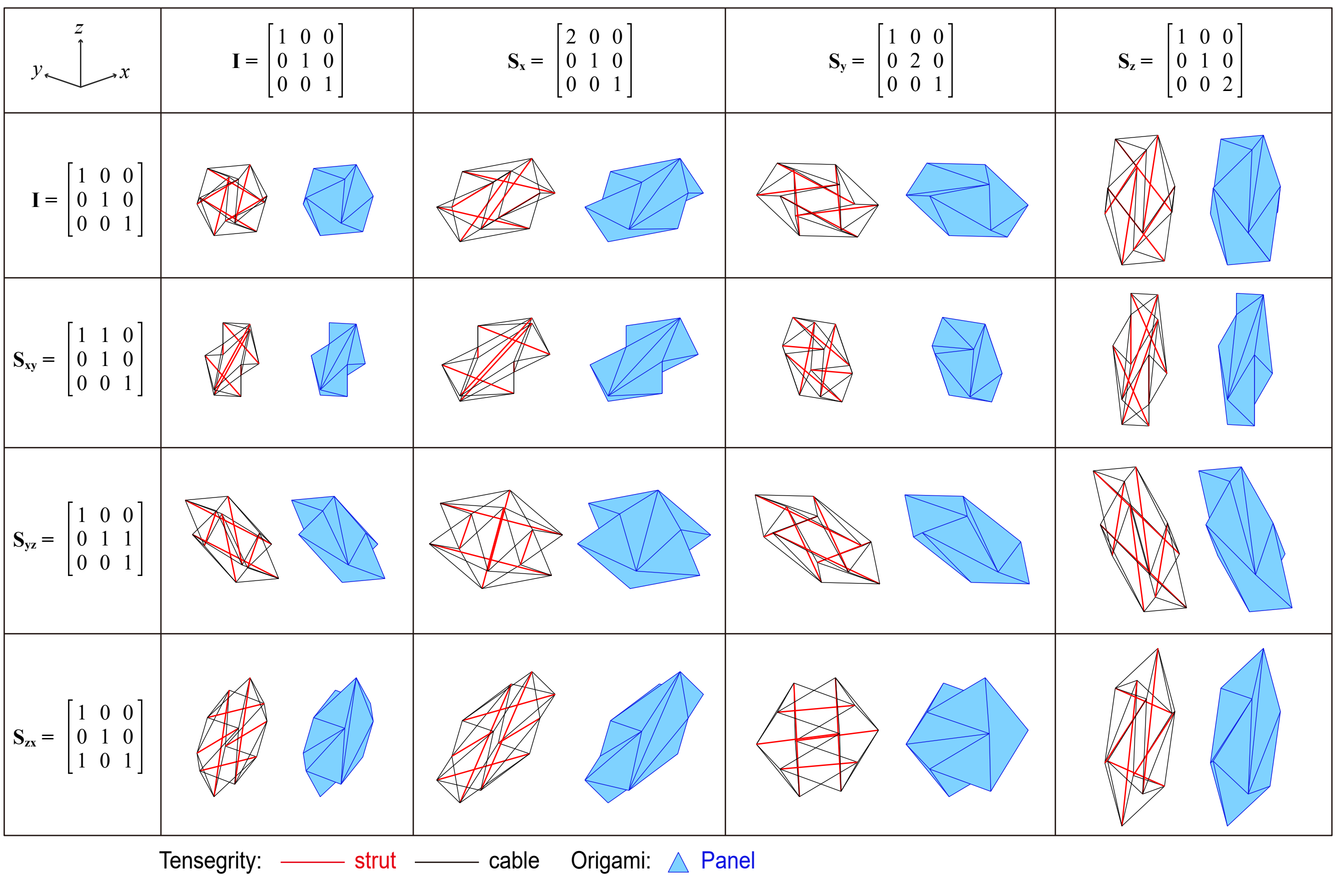

Fig. S10. Rhombic truncated tetrahedral tensegrities and the dual origami obtained through a linear transformation $\mathbf{T}$ applied to the regular configuration (Top Left). The transformation $\mathbf{T}$ can be an identity ($\mathbf{I}$), a stretching ($\mathbf{S}_{\mathrm{x}}$, $\mathbf{S}_{\mathrm{y}}$, $\mathbf{S}_{\mathrm{z}}$), a shear ($\mathbf{S}_{\mathrm{xy}}$, $\mathbf{S}_{\mathrm{yz}}$, $\mathbf{S}_{\mathrm{zx}}$), and their combinations ($\mathbf{S}_{\mathrm{xy}}\mathbf{S}_{\mathrm{z}}$, etc).

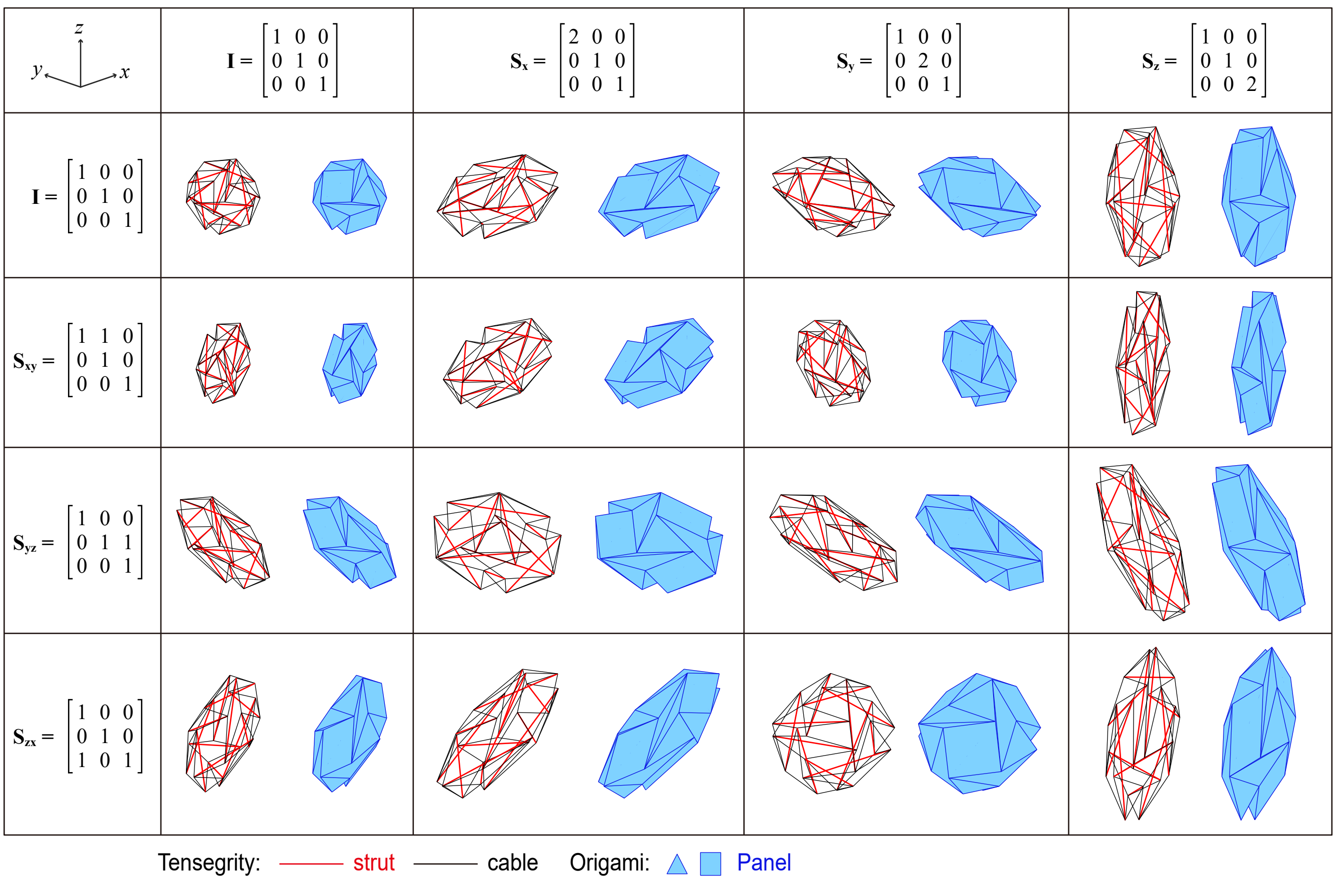

Fig. S11. Rhombic truncated cubic tensegrities and the dual origami obtained through a linear transformation $\mathbf{T}$ applied to the regular configuration (Top Left). The transformation $\mathbf{T}$ can be an identity ($\mathbf{I}$), a stretching ($\mathbf{S}_{\mathrm{x}}$, $\mathbf{S}_{\mathrm{y}}$, $\mathbf{S}_{\mathrm{z}}$), a shear ($\mathbf{S}_{\mathrm{xy}}$, $\mathbf{S}_{\mathrm{yz}}$, $\mathbf{S}_{\mathrm{zx}}$), and their combinations ($\mathbf{S}_{\mathrm{xy}}\mathbf{S}_{\mathrm{z}}$, etc).

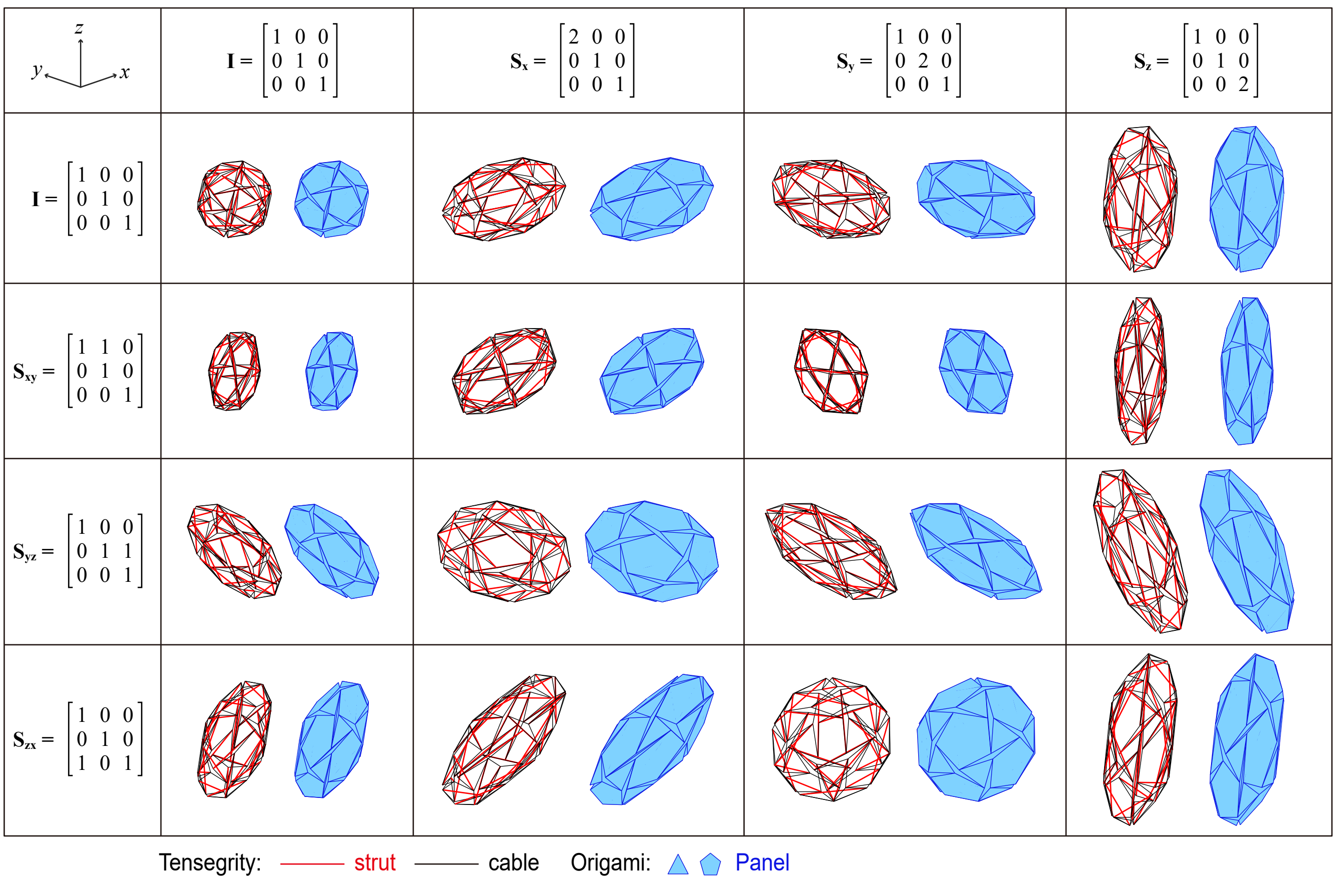

Fig. S12. Rhombic truncated dodecahedral tensegrities and the dual origami obtained through a linear transformation $\mathbf{T}$ applied to the regular configuration (Top Left). The transformation $\mathbf{T}$ can be an identity ($\mathbf{I}$), a stretching ($\mathbf{S}_x$, $\mathbf{S}_y$, $\mathbf{S}_z$), a shear ($\mathbf{S}_{xy}$, $\mathbf{S}_{yz}$, $\mathbf{S}_{zx}$), and their combinations ($\mathbf{S}_{xy}\mathbf{S}_z$, etc).

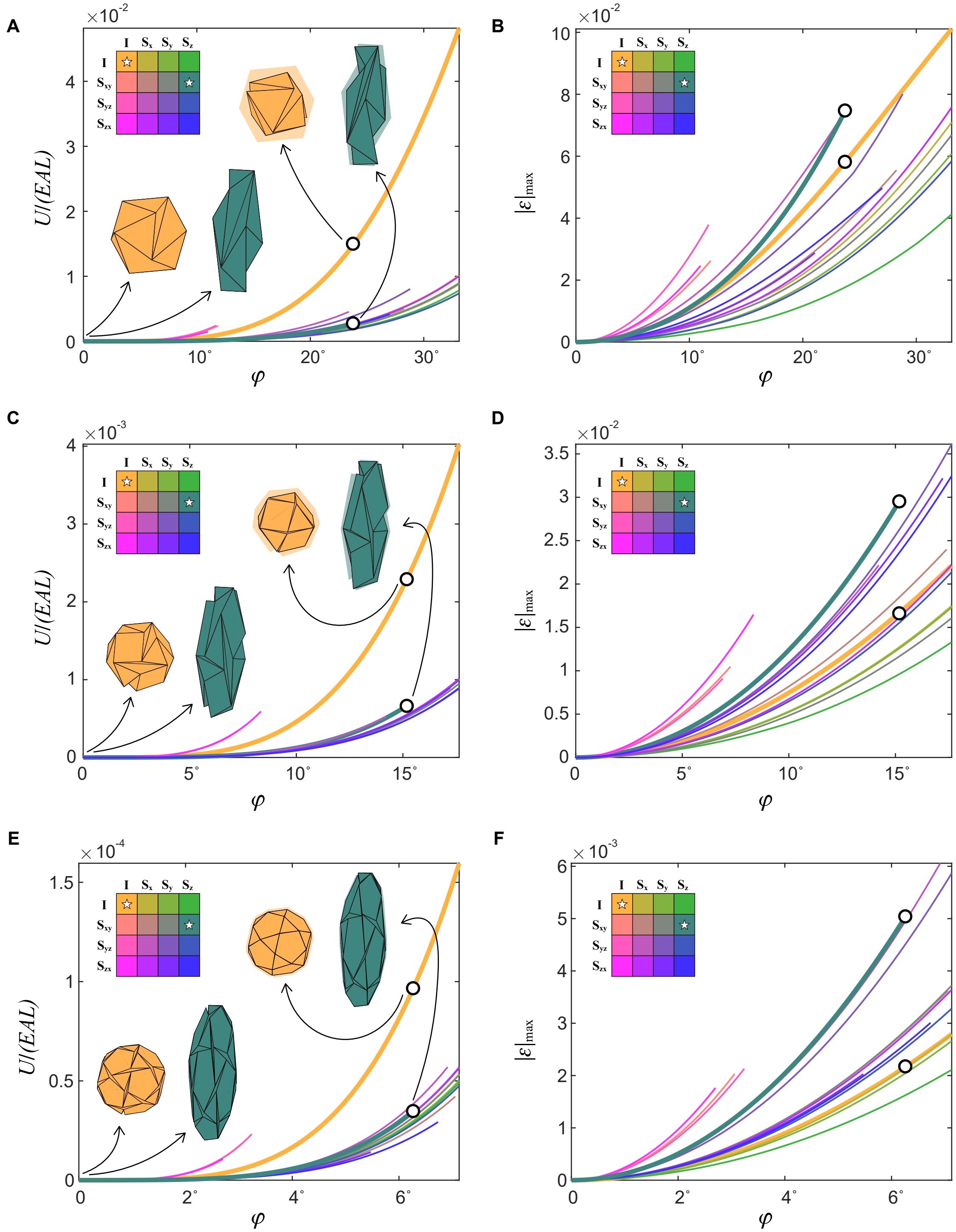

Fig. S13. Finite deformation simulation and approximation of shaky polyhedral origami. (A) The nondimensional energy $U/(EAL)$ and (B) the maximum edge strain $|\varepsilon|_{\max}$ of the origami obtained through $\mathbf{T}$ applied to the regular truncated tetrahedral configuration. The colormap is consistent with Fig. S10. (C) The nondimensional energy $U/(EAL)$ and (D) the maximum edge strain $|\varepsilon|_{\max}$ of the origami obtained through $\mathbf{T}$ applied to the regular truncated cubic configuration. The colormap is consistent with Fig. S11. (E) The nondimensional energy $U/(EAL)$ and (F) the maximum edge strain $|\varepsilon|_{\max}$ of the origami obtained through $\mathbf{T}$ applied to the regular truncated dodecahedral configuration. The colormap is consistent with Fig. S12. As illustrated by the colormaps, $\mathbf{T}$ can be an identity ($\mathbf{I}$), a stretching ($\mathbf{S}_{\mathrm{x}}$, $\mathbf{S}_{\mathrm{y}}$, $\mathbf{S}_{\mathrm{z}}$), a shear ($\mathbf{S}_{\mathrm{xy}}$, $\mathbf{S}_{\mathrm{yz}}$, $\mathbf{S}_{\mathrm{zx}}$), and their combinations ($\mathbf{S}_{\mathrm{xy}}\mathbf{S}_{\mathrm{z}}$, etc). For the regular origami, the simulations are performed under the principle of minimum potential energy. For the irregular origami ($\mathbf{T} \neq \mathbf{I}$), the finite nodal displacements are approximated with the linear formulation $\tilde{\mathbf{d}}_k = \det(\mathbf{T})\mathbf{T}^{-\mathrm{T}}\mathbf{d}_k$ (see Fig. 4 in the Main Text), yielding possible deformation paths that are used to generate the curves. The curves end when face contact or edge contact occurs.

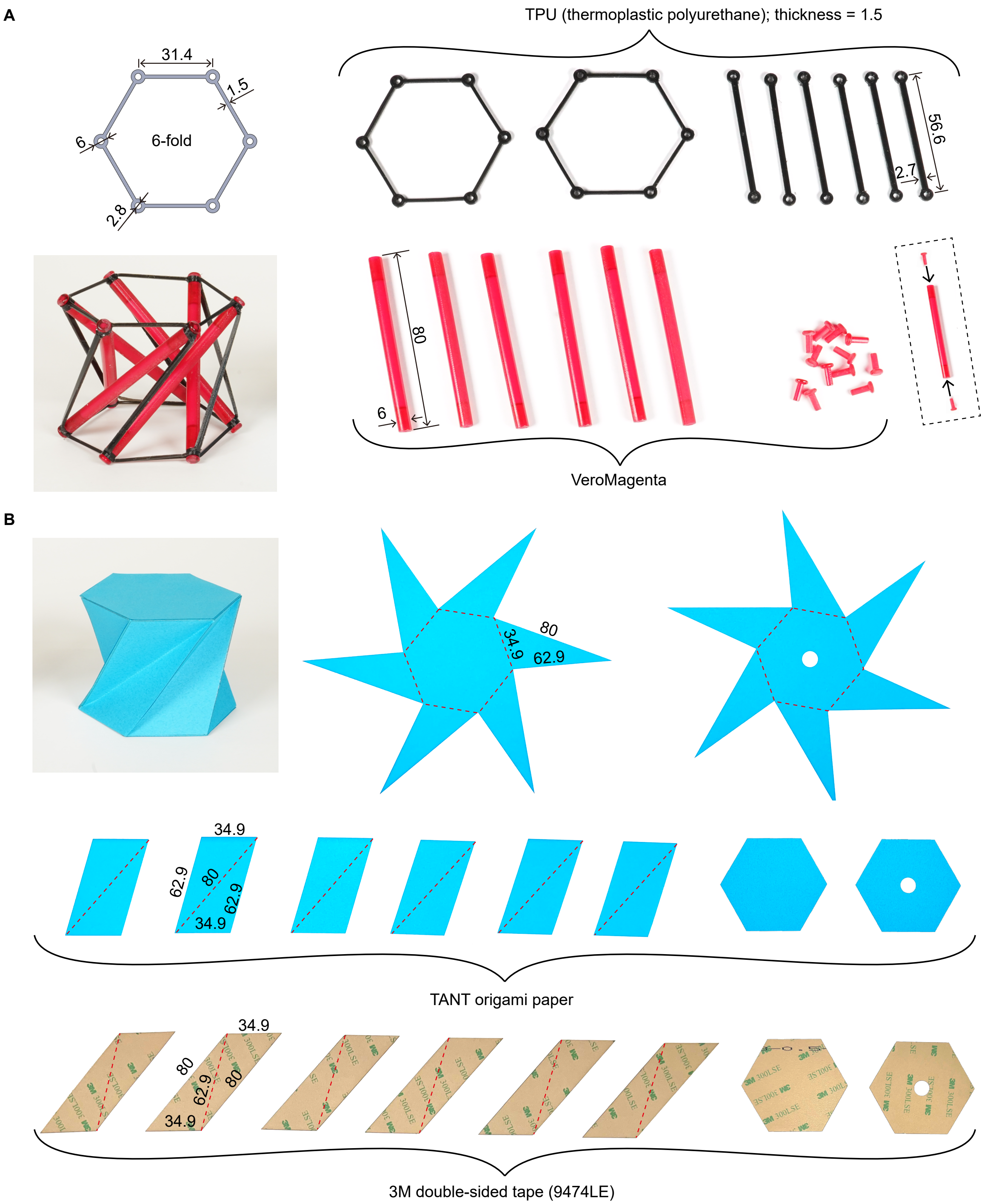

Fig. S14. Components of (A) the regular prismatic tensegrity and (B) the regular prismatic origami. All dimensions are in millimeters (mm).

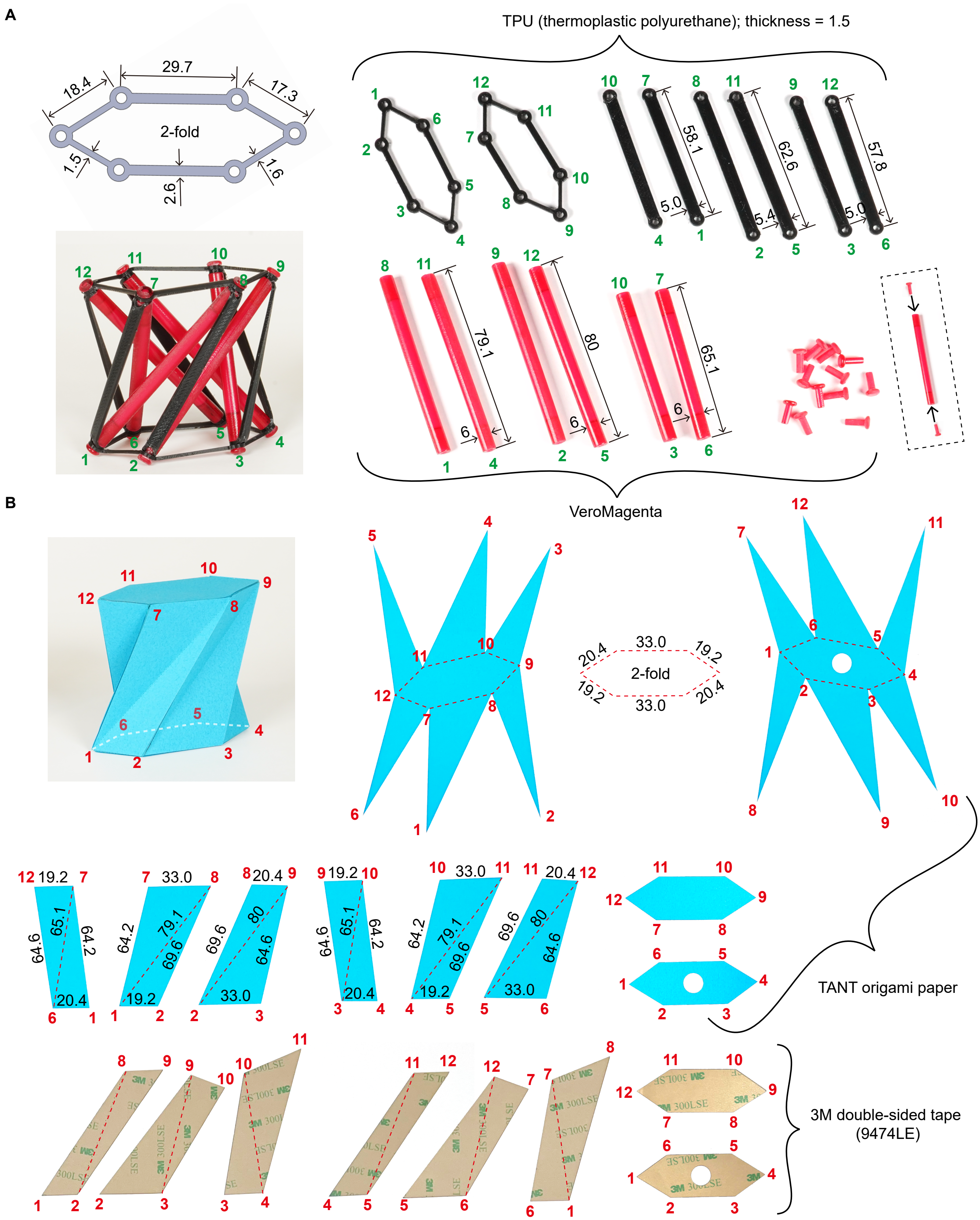

Fig. S15. Components of (A) the irregular prismatic tensegrity and (B) the irregular prismatic origami. All dimensions are in millimeters (mm).

Movie S1. Finite deformation simulation and approximation of shaky origami